MATH 363 Discrete Mathematics

SOLUTIONS : Assignment 3

1 Grading scheme

1. Full marks if stated $x = 1$ does not satisfy the inequality.
   -1pt If there is no computation of that fact.

2. Full marks if stated that $\sqrt[3]{2}$ being rational leads to a contradiction.
   -1pt if it is not clearly stated what the contradiction is,
   -1pt if statement 'n^3 being even implies n is even' is missing.

3. Full marks if argument includes the fact that perfect squares (which are positive) have difference at least 3.
   -pt if they don't justify this fact.
   Leave a note if the case 0^2 and 1^2 is not considered.

4. *i)* Full marks if they used the pigeonhole principle (implicitly or not)

   ii) Full marks if the proof is a paraphrase of the one below.
   1pt if they used the pigeonhole principle (implicitly or not) only once.

5. **+2pt** if they used the pigeonhole principle (implicitly or not)
   +1pt if they clearly state the 'boxes' and the 'pigeons'
   +1pt if they stated that 4 elements are not sufficient.

6. Full marks they show a valid tiling

7. Full marks if they paraphrase the proof below.
   -1pt If they state the pigeonhole principle is used in the proof.

8. **+1pt** For each item in the list as below. (A paraphrase in the last item suffices)

9. **+2pt** if basis step is correct,
   +2pt if inductive step is correct,
   -1pt if basis step is not clearly stated,
   -1pt if it is not clear where the inductive hypothesis is used.

10. **+2pt** if basis step is correct,
    +2pt if inductive step is correct,
    -1pt if basis step is not clearly stated,
    -1pt if it is not clear where the inductive hypothesis is used.

11. Full marks if stated that basis step is false.

2 Assignment with solutions

Some definitions to remember:
A non-negative integer n is a perfect square if there exists an integer k such that $n = k^2$.
The factorial numbers are defined as $n! = 1 \cdot 2 \cdot 3 \cdots n$ for all non-negative integers.

1. (**2pt**) Prove or disprove: if x is a nonzero real number, then $x^2 + 1/x^2 > 2$.
   Disprove: for $x = 1$ we don't have strict inequality: $(1)^2 + 1/(1)^2 = 1 + 1 = 2$.
   Note: For every real number x, if $x \neq 0$ then $(x - 1/x)$ is a well- defined real number. This implies that

$$\left(x - \frac{1}{x}\right)^2 \geq 0$$

   and thus $x^2 + \frac{1}{x^2} \geq 2$.

2. (**3pt**) Determine whether $\sqrt[3]{2}$ is rational.
   First note that if n is an odd integer (say $n = 2k+1$), then n^3 is also odd ($(2k+1)^3 = 2(4k^3+3k^2+3k)+1$).
   Now, suppose that $\sqrt[3]{2}$ is a rational number; that is, there are integers p, q with no common factors such that $\sqrt[3]{2} = p/q$. Then

$$2 = \frac{p^3}{q^3}$$

   and so $p^3 = 2q^3$ is even. We conclude that p is also even, (if p was odd, then p^3 would also be odd). Say $p = 2m$, then $8m^3 = 2q^3$ and thus, q^3 is even. Similarly, we have that q is even. But this is a contradiction to the assumption that p and q have no common factors. Therefore we conclude that $\sqrt[3]{2}$ is irrational.

3. (**4pt**) Prove that either $2 \cdot 3^{100} + 5$ or $2 \cdot 3^{100} + 6$ is not a perfect square.
   Let n and $n+1$ be a positive integers, we compute the difference between their squares:

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1 \geq 3.$$

   The last inequality follows from the assumption that $n \geq 1$. This implies that two consecutive perfect squares, which are both positive, differ by at least 3.
   Thus, since the difference between $a = 2 \cdot 3^{100} + 5$ and $b = 2 \cdot 3^{100} + 6$ is $|b - a| = 1 < 3$, then at least one of them is not a perfect square.

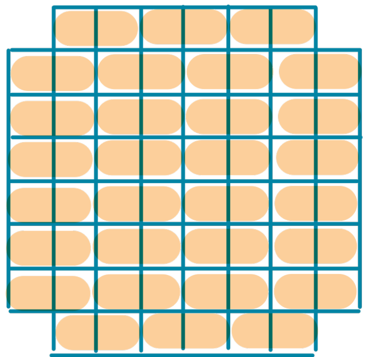
4. (**3pt each**) Consider a student with major on Software Engineering and minor in French literature that is taking 6 classes this term.

   - All her midterms are going to be on the week (Monday-Friday) before Reading week. Show that she is going to have a day with two midterms.
     There are 5 boxes (day of the week) in which we want to fit 6 pigeons (exams). By the pigeonhole principle, there is at least one day where the student will have 2 or more exams.
   - Show that she is taking either 3 classes that are all related or 3 classes that have no relation between each other.
     Suppose the classes she is taking are $c_1, \ldots, c_6$. Consider class c_6 there are other 5 classes which are either related to c_6 or not. By the pigeonhole principle, there are at least 3 pigeons (classes) which lie in the same box (either 'related to c_6' or 'not related c_6').
     Suppose w.l.o.g., that c_i is related to c_6 for each $i = \{1, 2, 3\}$. Now we consider two cases
     Case 1: c_i is **related** to c_j for some (distinct) $i, j \in \{1, 2, 3\}$. Then the classes c_i, c_j, c_6 are all related. As we wanted.
     Case 1: c_i is **not related** to c_j for all (distinct) $i, j \in \{1, 2, 3\}$. Then the classes c_1, c_2, c_3 are not related between each other.
     In either case, we find three classes that are all related or 3 classes that have no relation between each other.

5. (**4pt**) How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13\}$ to guarantee that at least one pair of these numbers add up to 14?
   First we partition the set $S = \{1, 3, 5, 7, 9, 11, 13\}$ into sets $\{1, 13\}$, $\{3, 11\}$, $\{5, 9\}$, $\{7\}$. By the pigeonhole principle, no matter how we select 5 elements in S, there is one set P of the partition which contains two of the 5 chosen elements. Clearly, $P \neq \{7\}$; but any other set of the partition contains precisely two numbers

which add up to 14.
Note that if we were to select only 4 elements in S, we have no guarantee that there is a pair of these numbers that add up to 14; for example when choosing $\{1, 3, 5, 7\}$

6. (**3pt**) Prove or disprove that you can use dominoes to tile the standard checkerboard with all four corners removed.
   Any example prove you can tile the checkerboard without the four corners



7. (**3pt**) Prove or disprove that you can use dominoes to tile the standard checkerboard with only two opposite corners removed.
   The checkerboard has $8 \times 8 = 64$ squares; there are 32 black squares and 32 white squares. When we remove two opposite corners, we are left with 62 squares: 30 squares of one colour and 32 of the other.
   Suppose that it is still possible to cover the checkerboard with dominos. We would need $62/2 = 31$ dominoes. Each domino covers one square of each colour, so at the end we would have 31 squares of each colour. This is a contradiction! There is a colour which has only 30 squares. So you cannot use dominoes to tile the standard checkerboard with two opposite corners removed.

8. (**1pt each**) Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for the positive integer n.

   - What is the statement $P(1)$? $P(1) : 1^3 = \frac{1^2(2)^2}{4}$
   - Show that $P(1)$ is true, completing the basis step of the proof. $P(1) : 1^3 = \frac{1^2(2)^2}{4} = 4/4 = 1$
   - What is the inductive hypothesis? That $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ holds.
   - What do you need to prove in the inductive hypothesis? If $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ then

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$$

   To see this, note that

$$\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^2\frac{4(n+1)}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

- Explain why these steps show that $\forall n \in \mathbb{Z}^+ P(n)$ is true.
  These are the steps of a proof by induction. It states that, knowing that $P(1)$ is true and, that for an arbitrary positive integer n we have that: $P(n)$ holds implies that $P(n+1)$, then it has to be the case that $P(n)$ is true for all positive integers.

9. (**4pt**) Use the same steps as in the previous example to prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

- The basis step is that $\frac{1}{2} = 1 - \frac{1}{1+1}$.
- This is true because the right-hand side of the equation is $1 - 1/2 = 1/2$.
- The hypothesis we will use is that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

  holds.
- Using the induction hypothesis we compute

$$\begin{aligned}\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} \\ &= 1 - \frac{1}{n+1} + \left(\frac{1}{(n+1)} - \frac{1}{n+2}\right) \\ &= 1 - \frac{1}{n+2}.\end{aligned}$$

  This completes the induction step, and thus the proof by induction.

10. (**4pt**) Prove using induction that $3^n < n!$ if n is a positive integer and $n > 6$.

- The basis step is that $3^7 < 7!$.
- This is true because $3^7 = 2187$ and $7! = 5040$.
- The hypothesis we will use is that $3^n < n!$ holds.(Note beside, $3^6 = 729$ and $6! = 720$)
- Using the induction hypothesis we compute

$$3^{n+1} = 3 \cdot 3^n < 3 \cdot n! < (n+1)!,$$

  the last inequality holds because $n > 6 \geq 3$. This completes the induction step, and thus the proof by induction.

11. (**2pt**) Explain what is wrong with the following proof:

- *Theorem*: For every positive integer n,

$$\sum_{i=1}^{n} i = \frac{(n+1/2)^2}{2}.$$

  Proof. We will prove it by induction: The formula is true for $n = 1$, this is the basis step. Now, suppose that

$$\sum_{i=1}^{n} i = \frac{(n+1/2)^2}{2},$$

Then, using the inductive hypothesis on the second equality,

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) \\ &= \frac{(n+1/2)^2}{2} + (n+1) \\ &= \frac{(n^2+n+1/4)+2(n+1)}{2} \\ &= \frac{n^2+3n+9/4}{2} \\ &= \frac{(n+3/2)^2}{2} \\ &= \frac{[(n+1)+1/2]^2}{2}.\end{aligned}$$

Thus, we have completed the inductive step and the theorem is true.

The inductive step is correct; if the formula is correct for n, then the formula is also correct for $n+1$. The problem lies in the fact that the formula is not correct for $n = 1$. That is, the base step is invalid.