MATH 363 Discrete Mathematics
Solutions: Assignment 4

1 Grading Scheme

1. Full marks if Diagram is correct.
2. Full marks if Bit string is correct.
3. There are several ways to prove it: For each question, full marks if answered.
   - Using Venn Diagrams. **-.5pt** if, besides the diagrams, there are no explanatory notes.
   - Using Membership Table. **-.5pt** if besides the table, there are no explanatory notes.
   - As shown below.
4. **+1pt** For each item answered. And Full marks if:

   i) Power set contains 16 elements.

   Leave a note if there are some parenthesis misplaced within the elements.

   ii) Stated: One cannot conclude anything.

   iii) Stated: The empty set is an element of C.
5. *i*) +3pt if they exhibit an element in $A \times B$ which is not contained in $B \times A$ (or viceversa). Similar as the solution below.

   +2pt if they only exhibit a particular example of distinct A and B where $A \times B \neq B \times A$. (e.g. $A = \{a, b\}$, $B = \{c, d\}$)

   Leave a note if they use matrix square brackets instead of (curly) brackets to place elements of the cartesian product.

   No marks if they performed product of vectors.

   ii) **+2pt** if bijection is $f(a, b) = (b, a)$.

   +1pt if there is a justification why this is a bijection.
6. Full marks if answer paraphrases solution below. Otherwise,

   +2pt if it justified the equality.

   +1pt if it considered only a particular case.
7. Full marks (each) if answers are correct.

   +1pt (each) if answered is wrong but some work is shown.
8. Full marks if it considers a general value $x \in \mathbf{Z}$ and $x \notin \mathbf{Z}$.

   +1pt if it considers only a particular value x for integers and non-integers.
9. Full marks if it explains how to find a injective function.

10. Full marks if it states a bijection is what we are looking for, and gives one.

    -1pt if function provided is not a bijection.

11. Full marks if answered yes.

12. **+1pt** for each correct answer (as shown below)

    -.5pt if a negative answered is not justified.

    Leave a note if the range is not expressed as a set.

13. Full marks if development of sum is displayed and correct.

    -1pt If there is a formula applied and there is no justification.

14. Full marks if development of sum is displayed and correct,

    -1pt If there is a wrong computation.

    -1pt If there is a wrong interpretation of the indices.

15. **+1pt** If example grows exponentially,

    +1pt If it is justified why the example is not injective.

16. Full marks if answered correctly

17. Full marks if answered correctly

18. **+1.5pt** If pair of witnesses C, k are good,

    +.5pt If explained how the inequality is satisfied with such witnesses.

19. **+2pt** If a correct algorithm is described.

    +1pt If pseudocode is presented.

20. **+2pt** If a correct bound using Big-O notation is stated,

    +1pt If such bound is explained.

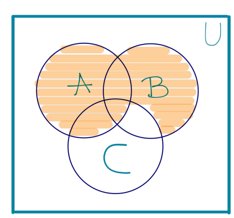
21. **+2pt** If a correct bound using Big-O notation is stated,

    +1pt If such bound is explained.

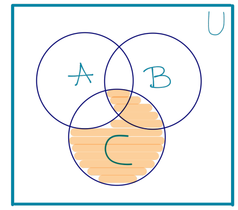
2 Assignment with solutions

1. (**1pt each**) Draw the Venn Diagram of the following sets.

   i) $[(A \cup B) \setminus C] \cup (A \cap C)$



ii) $(A^c \cup B) \cap C$



2. (**1pt each**) Let $U = \{x \in Z : x^2 < 10\}$, represent the follows sets using bit strings. Use the increasing order to list the elements in U. $U = \{-3, -2, -1, 0, 1, 2, 3\}$

i) $A = \{3, -2, 0, 1, -3\}$
1101101

ii) $B = \{x \in U,\ x > 0\}$
0000111

iii) $C = \{x \in U,\ |x - 1| < 2\}$
0001110

iv) $(A \cap C)$
0001100

v) $(A \cup B)$
1101111

vi) $[(A \cup B) \setminus C] \cup (A \cap C)$
1101101

3. (**2pt each**) Let A, B be sets. Show that

i) $(A \cap B) \subseteq (A \cup B)$
If $x \in A \cap B$ then $x \in A$ and $x \in B$. In particular, $x \in A$. Thus, $x \in (A \cup B)$.

ii) $A \cup B = A \cup (B \setminus A)$
First we will show that $A \cup B \subseteq A \cup (B \setminus A)$. Let $x \in A \cup B$, then $x \in A$ or $x \in B$. Now, we will prove that $x \in A \cup (B \setminus A)$ by considering two cases:
Case 1: $x \in A$. Thus, $x \in A \cup (B \setminus A)$.
Case 2: $x \notin A$. Then it must be the case that $x \in B$. Since $x \in B$ and $x \notin A$, we have $x \in B \setminus A$. Therefore $x \in A \cup (B \setminus A)$.
Second, we will show that $A \cup (B \setminus A) \subseteq A \cup B$. Let $x \in A \cup (B \setminus A)$, then $x \in A$ or $x \in B \setminus A$. Now, we will prove that $x \in A \cup B$ by considering two cases:
Case 1: $x \in A$. Thus, $x \in A \cup B$.
Case 2: $x \in (B \setminus A)$. In particular, this means $x \in B$ and thus $x \in (A \cup B)$.

4. (**2pt** each) Let $A = \{\emptyset, a, b, \{a, \{b\}\}\}$

i) List all the elements of $B = \mathcal{P}(A)$
$B = \{\ \emptyset\ ,\ \{a\}, \{b\}, \{\{a, \{b\}\}\}\ ,\ \{a, b\}, \{a, \{a, \{b\}\}\}, \{b, \{a, \{b\}\}\}\ ,\ \{a, b, \{a, \{b\}\}\}$
$\{\emptyset\}, \{\emptyset, a\}, \{\emptyset, b\}, \{\emptyset, \{a, \{b\}\}\}\ ,\ \{\emptyset, a, b\}, \{\emptyset, a, \{a, \{b\}\}\}, \{\emptyset, b, \{a, \{b\}\}\}\ ,\ \{\emptyset, a, b, \{a, \{b\}\}\}\}$

ii) If $\{\emptyset\} \subset \mathcal{P}(C)$, what can you conclude about C? Justify your answer.
One cannot conclude anything; this is a fact that occurs for all sets C. Recall that $\emptyset \subset C$, for any set C. Therefore, $\emptyset \in \mathcal{P}(C)$ and so $\{\emptyset\} \subset \mathcal{P}(C)$.

iii) If $\{\emptyset\} \in \mathcal{P}(C)$, what can you conclude about C? Justify your answer.
If there is a singleton S in $\mathcal{P}(C)$ then the element contained in S is an element of C. Thus, we can conclude that $\emptyset \in C$ (the empty set is **an element** of C).

5. (**3pt each**) Let A and B be distinct non-empty sets

i) Show that $A \times B \neq B \times A$,
If $A \neq B$, let's suppose without loss of generality that there exists an element $b \in B$ such that $b \notin A$. Since A is non-empty, there exists at least one element $a \in A$.
Now, the ordered pair $(a, b) \in A \times B$ but our choice of b implies that $(a, b) \notin B \times A$. We have shown that there is at least one element in $A \times B$ which is not contained in $B \times A$, so $A \times B \neq B \times A$.

ii) Give a bijection between $A \times B$ and $B \times A$.
Let $f : (A \times B) \to (B \times A)$ be defined as $f(a, b) = (b, a)$.
Note that f is well define beacause if $(a, b) \in A \times B$ that means that $a \in A$ and $b \in B$ and then, clearly, $(b, a) \in B \times A$.
First we show that f is surjective: Consider a pair $(x, y) \in B \times A$; with a similar argument as above, we have that $(y, x) \in A \times B$, and furthermore $f(y, x) = (x, y)$; as desired.
Second we show that f is injective: We have to prove that for any two distinct elements (a_1, b_1), $(a_2, b_2) \in A \times B$ we have $f(a_1, b_1) \neq f(a_2, b_2)$. This is proven directly: the assumption is that $a_i \neq b_i$ for $i = 1$ or $i = 2$. W.l.o.g, let us assume that $a_1 \neq b_1$. Then the second coordinates of $f(a_1, b_1)$, $f(a_2, b_2)$ are distinct, which is what we wanted to prove.
Since f is both injective and surjective, then f is a bijection from $A \times B$ to $B \times A$

6. (**3pt**) Show that if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$
We will show that under the conditions above, $A \subseteq B$ and $B \subseteq A$:
First, consider $x \in A$, we will analyse two cases.
Case 1: If $x \in C$, then $x \in A \cap C$, by the hypothesis $A \cap C = B \cap C$, we have that $x \in B \cap C$ and in particular $x \in B$.
Case 2: If $x \notin C$. Since we started with $x \in A$ we have $x \in A \cup C$, by the hypothesis $A \cup C = B \cup C$, we have that either $x \in B$ or $x \in C$. The latter can not occur (this is Case 2), so it must be the case that $x \in B$.
In either case, we showed that $x \in A$ implies $x \in B$. Thus, $A \subseteq B$. The proof that $B \subseteq A$ is the same as above, by interchanging the roles of A and B.

7. (**2pt** each) Let $A_i = \{1, 2, \dots, i\}$. Give a description of the following sets.

i) $A_{2i} \setminus A_{2i-1}$

$= \{2i\}$

ii) $\bigcap_{i=1}^{40} (A_{2i} \setminus A_{2i-1})$

$= \emptyset$

iii) $\bigcap_{j=3i-1}^{3i+1} A_j$

$= A_{3i-1}$

iv) $\bigcup_{i=2}^{4} \left(\bigcap_{j=3i-1}^{3i+1} A_j \right)$

$= A_{11}$

8. (**2pt**) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \lceil x \rceil - \lfloor x \rfloor$. Show that $f(x) = 0$ if $x \in \mathbf{Z}$ and $f(x) = 1$ if $x \notin \mathbf{Z}$.
It is clear that if x is an integer, then $\lceil x \rceil = \lfloor x \rfloor = x$. Thus $f(x) = 0$.
Now, consider a non-integer real number x; we can write it as $x = k + \varepsilon$ where $k \in \mathbf{Z}$ and $\varepsilon \in (0, 1)$. In this case, $\lfloor x \rfloor = k$ and $\lceil x \rceil = k + 1$. Thus, $f(x) = (k + 1) - k = 1$.

9. (**2pt**) Prove that for any non-empty countable set S, there exists a injective function from S to $\mathbf{N}$.
There are two types of countable sets: finite, or infinite. By the definition used in class, S is countable and infinite when there exists a bijective function $f : S \to \mathbf{N}$. In particular, this function f is injective.
Now, if S is finite, there exists a bijective function $f : S \to \{1, \dots, n\}$, for some $n \in \mathbf{N}$. Let $g : S \to \mathbf{N}$ be defined as $g(s) = f(s)$ for all $s \in S$. Then it follows that g is injective; otherwise there are two distinct elements $s, t \in S$ such that $g(s) = g(t)$. This would lead to $f(s) = f(t)$ which is a contradiction because f is a bijection.

10. (**3pt**) Show that the set of points in the circle $C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ have the same cardinality that the points in the interval $I = [0, 1)$.
It suffices to give a bijection between I and C; for example
$f : I \to C$, with $f(x) = (\cos(2x\pi), \sin(2x\pi))$
$g : C \to I$, with $f(x, y) = \frac{\arctan(x/y)}{2\pi}$

11. (**1pt**) Let P be the set containing the 5 pythagorean solids. Is P countable?
Yes. (The set is $P = \{\text{tetrahedron, cube, octahedron, dodecahedron, icosahedron}\}$)

12. (**2pt each**) Let $f, g, h : P \to \mathbf{N}$ be functions such that, if $p \in P$ then

$$\begin{aligned} f(p) &= n && \text{if } p \text{ has } n \text{ faces} \\ g(p) &= m && \text{if } p \text{ has } m \text{ edges} \\ h(p) &= v && \text{if } p \text{ has } v \text{ vertices} \end{aligned}$$

For example, $f(\text{cube}) = 6$, $g(\text{cube}) = 12$, $h(\text{cube}) = 8$. (Justify your answers)

i) Are the functions f and g injective?
f is injective: $f(\text{tetrah.}) = 4$, $f(\text{cube}) = 6$, $f(\text{octah.}) = 8$, $f(\text{dodecah.}) = 12$, $f(\text{icosah.}) = 20$
g is not injective: $g(\text{cube}) = g(\text{octah.}) = 12$ and $g(\text{dodecah.}) = g(\text{icosah.}) = 30$

ii) Are the functions f and g surjective?
Both are not surjective since there is no preimagen for $10 \in \mathbf{N}$, for example.

iii) What is the range of g and $f + h$?
$g(P) = \{6, 12, 30\}$ and $(f + h)(P) = \{8, 14, 32\}$

13. (**2pt** each) Find the value of the following sums. Show your work.

i) $\displaystyle\sum_{i=1}^{40}(3 + 5i)$

$$= \left(\sum_{i=1}^{40} 3\right) + 5\sum_{i=1}^{40} i$$
$$= (3 \cdot 40) + 5\frac{(40)(41)}{2}$$
$$= 120 + 4100 = 4220$$

ii) $\displaystyle\sum_{i=0}^{10}(1/3)^i = \frac{1 - (1/3)^{11}}{(1 - 1/3)}$

or working out the formula

$$= \frac{(1 - 1/3)\sum_{i=0}^{10}(1/3)^i}{(1 - 1/3)}$$
$$= \frac{\left(\sum_{i=0}^{10}(1/3)^i\right) - \left(\sum_{i=1}^{11}(1/3)^i\right)}{\frac{2}{3}}$$
$$= \frac{1 - \frac{1}{3^{11}}}{\frac{2}{3}} = \frac{3^{11} - 1}{2 \cdot 3^{10}}$$

14. (**3pt** each) Find the value of the following sums. Show your work.

i) $\displaystyle\sum_{i=2}^{4}\sum_{j=1}^{2i-3} 2j$

$$= \sum_{i=2}^{4} 2\left(\sum_{j=1}^{2i-3} j\right)$$
$$= \sum_{i=2}^{4} 2\left(\frac{(2i-3)(2i-2)}{2}\right)$$
$$= \sum_{i=2}^{4}(2i-3)(2i-2)$$
$$= (1 \cdot 2) + (3 \cdot 4) + (5 \cdot 6) = 44$$

ii) $\displaystyle\sum_{i=2}^{4}\sum_{j=1}^{i}\lfloor 3j/2 \rfloor$

$$= \sum_{j=1}^{2}\lfloor 3j/2\rfloor + \sum_{j=1}^{3}\lfloor 3j/2\rfloor + \sum_{j=1}^{4}\lfloor 3j/2\rfloor$$
$$= 3\left(\lfloor 3 \cdot 1/2\rfloor + \lfloor 3 \cdot 2/2\rfloor\right)$$
$$+2\left(\lfloor 3 \cdot 3/2\rfloor\right) + \left(\lfloor 3 \cdot 4/2\rfloor\right)$$
$$= 3(1 + 3) + 2(4) + 6 = 26$$

15. (**2pt**) Give an example of a function from $\mathbf{N} \to \mathbf{N}$ which is not injective and it grows at least exponentially. Justify your answer.

Let $f(n) = (\lceil (n+1)/2 \rceil)^{\lceil (n+1)/2 \rceil}$. To see that f is not injective, note that $f(2) = f(3)$. This follows since $f(2) = (\lceil 3/2 \rceil)^{\lceil 3/2 \rceil} = 2^2$ and $f(3) = (\lceil 4/2 \rceil)^{\lceil 4/2 \rceil} = 2^2$.
That f grows at least exponential is the same as showing that f is $\Omega(e^n)$.
This in turn means that for some positive constants c, a, K we have $|f(n)| \geq c \cdot a^n$ for $n \geq K$.
Note that $\lceil (n+1)/2 \rceil \geq n/2 \geq 4$ for all $n \geq 8$. Thus, for all $n \geq 8$
$|f(n)| = f(n) \geq \left(\frac{n}{2}\right)^{n/2} \geq \left(\sqrt{n/2}\right)^n \geq 2^n$.

16. (**1pt each**) Determine whether the following statements are true.

    i) $2 + x^3$ is $O(x^2)$ False

    ii) $2 + x^3$ is $\Omega(x^2)$ True

    iii) $\lfloor 2x \rfloor$ is $O(x)$ True

17. (**1pt each**) Give a big-O estimate for the following functions

    i) $(n^3 + n^2 \log n)(3 - 2\log n)$ is $O(n^4)$ and is also $O(n^3 \log n)$

    ii) $(n! - n^5)(3^n + 5^n)$ is $O(n! \cdot 5^n)$ and is also $O((5n)^n)$, the latter uses Stirling formula

18. (**2pt each**) Use the definition of '$f(x)$ is $O(g(x))$, $\Omega(g(x))$, $\Theta(g(x))$' to show that

    i) $n^3 \log(n^2)$ is $\Omega(n^3)$,
    By definition this means that there are positive constants c, K such that

    $$|n^3 \log(n^2)| \geq cn^3 \text{ for all } n \geq K.$$

    That is true if we set $c = 1$ and $K = 2$. This follows since $\log(n^2) \geq 1$ if $n \geq 2$.

    ii) $\log(3^n)$ is $O(n)$,
    By definition this means that there are positive constants c, K such that

    $$|\log(3^n)| \leq cn \text{ for all } n \geq K.$$

    That is true if we set $c = \log 3$ and $K = 1$. This follows since $\log(3^n) = n \log 3$.

    iii) $n^3 + 5n^2 + 40$ is $\Theta(n^3)$.
    By definition this means that there are positive constants c_1, c_2 and K such that

    $$c_1 n^3 \leq |n^3 + 5n^2 + 40| \leq c_2 n^3 \text{ for all } n \geq K.$$

    That is true if we set $c_1 = 1$, $c_2 = 7$ and $K = 4$.
    Consider first the upper bound. For all $n \geq 4$ we have $n^3 \geq 64$, thus

    $$|n^3 + 5n^2 + 40| = n^3 + 5n^2 + 40 \leq 6n^3 + 40 \leq 7n^3.$$

    The lower bound is straightforward, for n positive: $|n^3 + 5n^2 + 40| = n^3 + 5n^2 + 40 > n^3$.

19. (**2pt**) Describe an algorithm for finding both the largest and the smallest integers in a finite sequence of integers.

20. (**3pt**) Determine the worst-case performance of the algorithm above.

21. (**3pt**) Determine the least number of comparisons, or best-case performance required to find the maximum of a sequence of n integers (using the algorithm from class).