MATH 363 Discrete Mathematics
SOLUTIONS: Assignment 7

1 Grading Scheme

1. Out of **2pt**: +1pt If the basis step is stated and correct.

   +1pt If the inductive step is stated and correct.

2. Out of **3pt**: Full marks if an algorithm is given.
   -1pt if it is unclear or has minor mistakes.

3. Out of **3pt**: +2pt if the recursive formula is given
   +1pt if a clear justification of the formula is given

4. Out of **2pt**: +1pt if the big-O estimate is correct
   +pt if it is cleary justified with the Master theorem or an equivalent argument. **Leave a note:** explaining why the conditions for the Master theorem are satisfied, if not stated why.

2 Assignment with solutions

1. (**2pt**) Let $\alpha = \frac{1+\sqrt{5}}{2}$. Recall that the first Fibonacci numbers are $f_0 = 0$, $f_1 = 1$, $f_2 = 1$. Use induction to prove that for any integer $n \geq 3$, f_n satisfies

   $$f_n > \alpha^{n-2}$$

   The basis step starts by verifying that $f_3 = 2 > \alpha$. This holds since $1 + \sqrt{5} < 1 + 3$.
   For the induction step we assume that $f_k > \alpha^{k-2}$ for all $k < n$. Then

   $$f_n = f_{n-1} + f_{n-2} > \alpha^{n-3} + \alpha^{n-4} = (\alpha + 1)\alpha^{n-4} = \alpha^{n-2}.$$

   The last equation follows from the fact that $1 + \alpha = \frac{3+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = \alpha^2$.

2. (**3pt each**) Give the pseudocode of a recursive algorithm to:

   - compute the greatest common divisor of two distinct positive integers a, b.
     procedure $gcd(a, b:$ integers $0 \leq a < b)$
     if $a = 0$ **then** $gcd(a, b) := b$
     else $gcd(a, b) := gcd(b\,\mathbf{mod}\,a, a)$

   - give a factorization into prime factors of a positive integer n.
     procedure $fac(n:$ integer $2 \leq n)$ is an array with a factorization of n into primes
     f:=[ ]
     $M = \lfloor \sqrt{n} \rfloor$
     k:=2
     s:=0
     while $s := 0$
     while $k := 2 \leq M$

if k divides n **then**
$fac(n)$:=append $fac(k)$ to $fac(n/k)$
$s := 1$
else $k := k + 1$

3. (**3pt**) Let z_n denote the number of zero-one strings of length n which do not contain two consecutive zeros. Find a recursive formula for z_n and justify this formula.
Let's call *valid* to a bit string which does not contain two consecutive zeros. A valid string s of length n either start with 0 or with 1. If s it start with 1, then deleting this bit will uniquely determine a valid string of length $n - 1$. If s starts with 0, then the second bit is 1; deleting these two bits will uniquely determine a valid string of length $n - 2$.
We just argued that the set of valid strings of length $n-1$ are in one-to-one correspondence with valid strings of length n starting with 1. And the set of valid strings of length $n - 2$ are in one-to-one correspondence with valid strings of length n starting with 0. Thus,

$$z_n = z_{n-1} + z_{n-2}.$$

4. (**2pt** each) Let the functions f, g satisfy the following recursive relation:

   - $f(n) = 9f(n/3) + 20n^2$ when $n = 3k$,
   Using the notation of the master theorem we have $a = 9$, $b = 3$, $c = 20$, $d = 2$.
   The condition on the recursive formula is satisfied for all n multiple of $b = 3$, in particular it is satisfied for all n which is a power of b.
   Since $a = 9 = b^d$, we have that $f(n)$ is $O(n^2 \log n)$.
   - $g(n) = 16g(n/4) + n$ $n = 4k$.
   Using the notation of the master theorem we have $a = 16$, $b = 4$, $c = 1$, $d = 1$.
   The condition on the recursive formula is satisfied for all n multiple of $b = 4$, in particular it is satisfied for all n which is a power of b.
   Since $a = 16 > 4 = b^d$ and $\log_4 16 = 2$, we have that $f(n)$ is $O(n^2)$.

   Give a big-O estimate for f and g.