MATH 363 Discrete Mathematics
Assignment 8

Due by March 16th

# 1 Grading Scheme

For questions 1-4 and 9-10:

1. Full marks if the numerical expression is correct
2. Full marks if a numerical expression is given together with an argument justifying it.
3. -1pts if an incorrect numerical expression is given together with an argument justifying it. (Leave a note on the first invalid argument used).
4. 0pts if only a numerical expression is given and it is incorrect.

For combinatorial proofs, questions 5*ii)*,6-7, 8*ii)*. For each of the expressions in the equality:

1. 1pt if a valid justification is given
2. .5pt if a justification is given but is incorrect/incomplete. (Leave a note on the first invalid idea used).

For questions 5*i)* and 8*i)*

1. Full marks if an answer paraphrasing answers below are given.
2. Half marks if an answer is given but is invalid. (Leave a note on the first invalid idea used).

# 2 Assignment with solutions

1. (**2pt**) How many bit strings of length 10 either start with the bits 101 or end with the bits 111?
   $2^7 + 2^7 - 2^4$
   First we count the number of bit strings $101xxxxxxx$ where each of the $x$ can take the value 0 or 1, thus there are $2^7$ possibilities. Similarly, there are $2^7$ bit strings of the form $xxxxxxx111$. These two cases overlap and we have to substract those in our count. There are $2^4$ bit strings of the form $101xxxx111$.

2. (**3pt**) How many bit strings of length 10 contains 5 consecutive zeros?
   **If want exactly 5 consecutive zeros:** $64 = 2^4 + 2^3 + 2^3 + 2^3 + 2^3 + 2^4$
   The 6 possible forms of the bit strings are:
   $000001xxxx$
   $1000001xxx$
   $x1000001xx$
   $xx1000001x$
   $xxx1000001$
   $xxxx100000$
   **If want at least 5 consecutive zeros: 112**
   What we have to note here is that this constraint can be divided into disjoint cases:
   'bit strings of length 10 with exactly $k$ consecutive zeros' for $5 \leq k \leq 10$.
   Following the idea above, there are 64 bit strings for $k = 5$,

there are $2^3 + 2^2 + 2^2 + 2^2 + 2^3 = 28$ for $k = 6$, (e.g. $0000001xxx, 10000001xx$),
there are $2^2 + 2 + 2 + 2^2 = 12$ for $k = 7$, (e.g. $00000001xx, 100000001x$),
and finally, there are $2 + 1 + 2 = 5$ for $k = 8$, 2 for $k = 9$ and 1 for $k = 10$.

3. (**2pt**) How many bit strings of length 10 contains either 5 consecutive zeros or 5 consecutive ones?
$112 + 112 - 2 = 222$
By symmetry, there are 112 bit strings of length 10 containing 5 consecutive ones. We have to subtract the two bit strings that contain both 5 consecutive zeros and 5 consecutive ones.

4. (**2pt**) A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?
$\binom{40}{17}$
To see this, note that the answer key is just a sequence of 40 characters $T, F$ with exactly 17 $T$'s.

5. (**2pt each**) Prove $2^n = \sum_{j=0}^{n} \binom{n}{j}$

   *i*) Using the binomial theorem
   Apply the binomial theorem for $x = y = 1$. Then $(1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^j 1^{n-j}$. Which simplified gives the desired equality.

   *ii*) Using a combinatorial proof
   Consider a set $S = \{s_1, \dots, s_n\}$. We will count the total number of subsets $B \subseteq S$.
   First, we can use the bit string representation of sets using $S$ as our universe. Then each subset $B$ correspond to a bit string of length $n$; we know that the are $2^n$ such bit strings.
   Another way of counting the subsets $B$ is by separating them according to the cardinality of $B$. For any $0 \le j \le n$, if we want that $|B| = j$ then there are $\binom{n}{j}$ possible sets.

6. (**2pt**) Let $n, m \ge k \ge 1$ be integers. Using a combinatorial proof, prove that

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}.$$

   **Hint:** Pick a comittee of a given size, out of a group of men and women.
   One way of pick a committee of $k$ people out of a total of $m + n$ men and women is just to select $k$ people out of a group of $m + n$. So there are $\binom{m+n}{k}$ ways to do it.
   Another way is to determine first, how many men are going to be part of the committee. If there are $j$ men, then there are $n - j$ women and the choices can be done separately. So in total, there are $\binom{m}{j}\binom{n}{k-j}$ ways to select a committee with $k$ men and $k - j$ women. Finally, $0 \le j \le k$ and these choices do not overlap.

7. (**2pt**) Let $n, k \ge 1$ be integers. Using a combinatorial proof, prove that

$$\binom{n+k+1}{k} = \sum_{j=0}^{k} \binom{n+j}{j}.$$

   **Hint:** Consider bit strings with a fixed number of zeros. Then divide in cases according to the place of the last bit with a one.
   One way to construct a bit string with $k$ zeros and $n + 1$ ones is to select the location of the $k$ zeros out of a bit string of length $n + k + 1$. There are $\binom{n+k+1}{k}$ ways of doing that.
   Another way is to separate into cases according to the position of the last 'one' of the bit string. If the last 'one' is in position $m$, the bit string will have the form: $xx \cdots xx100 \cdots 00$; where the bit string $xx \cdots xx1$ has length $m$ and exactly $n + 1$ 'ones'.
   Now, since there are $n + 1$ 'ones' in the original bit string, then $m$ satisfies $n + 1 \le m = n + 1 + j$ with $0 \le j \le k$. The number of ways to place the remaining 'ones' into the first part $xx \cdots x100 \cdots 00$ is $\binom{n+j}{n} = \binom{n+j}{j}$.

8. (**2pt each**) Let $m, n \geq 1$ be integers. Consider paths in a grid connecting the points $(0,0)$ and $(m,n)$ made out of steps either one unit to the right or one unit upward (No moves to the left or downward are allowed).

   *i*) Show that there are $\binom{n+m}{n}$ distinct such paths.
   (**Hint:** Represent each path as a bit string encoding the steps going either up or to the right.)
   Each path can be encoded with a list of $m+n$ characters $R, U$ that mean either 'right' or 'up'. We have to take exactly $m$ steps to the right (and thus, $n$ steps up). So the number of possible paths are $\binom{m+n}{m}$.

   *ii*) Use this experiment to prove Pascal's identity.
   (**Hint:** Consider that each path either passes through (0,1) or (1,0).)
   Suppose we have $k > l \geq 0$ integers and we want to count the number of paths in the grid that connect $(0,0)$ to $(l+1, k-l)$ as above. We have seen that there are $\binom{(k-l)+l+1}{l+1} = \binom{k+1}{l+1}$ such paths.
   Since the first step of the path can be either up or to the right we can count the number of different paths with that property. If the first step is $R$, then it remains to encode $k$ steps with $l$ of them to the right: $\binom{k}{l}$. If the first step is $U$, then it remains to encode $k$ steps with $l+1$ of them to the right: $\binom{k}{l+1}$. This gives pascal's identity,

$$\binom{k+1}{l+1} = \binom{k}{l+1} + \binom{k}{l}.$$

9. (**3pt**) Maria will read a book with 16 chapters during her free time over 5 weekends, she will read in order and at least one chapter per week. In how many ways can she read the book? (e.g. Week 1 Chapters 1-4, Week 2 Chapters 5-10, Week 3 Chapter 11, Week 4 Chapters 12-15, Week 5 Chapter 16)
   $\binom{15}{4}$
   **Solution 1:** Suppose that Maria will place 4 bookmarks to know when to stop reading at each of the first 4 weekend; she doesn't need a bookmark to stop reading at the last weekend because she will reach to the end of the book.
   In this case, there are 15 spaces between consecutive chapters; and we want to place exactly one bookmark in 4 of these 15 spaces. Thus, there are $\binom{15}{4}$ ways to read the book. (Note that the constraint of placing no more than 1 bookmark per space is equivalent to say that Maria will read at least one chapter per weekend.)
   **Solution 2:** We would like to apply the bar-and-stars trick. In which we select the number of chapters read at weekend $w_i$, $1 \leq i \leq 5$. So that $w_1 + \cdots + w_6 = 16$. (In the slides we saw the cases where each $w_i$ represented a different kind of bill.) Now we have a different constraint, which is that Maria has to read at least one chapter every week; and the bars-and-stars trick would allow for some $w_i = 0$. The way we can solve it is, informally, to 'put side' 5 chapters. And, thus, to use the trick for 11 chapters and 5 weeks: $\binom{11+(5-1)}{(5-1)} = \binom{15}{4}$ ways of reading the book.
   More formally, let $s_i = w_i - 1$ be the number of 'extra' chapters Maria will read during the $i$-th weekend. Then $w_1 + \cdots + w_5 = 16$ with $w_i \geq 1$, is equivalent to $s_1 + \cdots s_5 = 11$ with $s_1 \geq 0$. And we can apply the bar-and-stars trick directly.

10. (**2pt each**) A coin is flipped 10 times (it lands either tails or heads with equal probability)

   *i*) How many possible outcomes are there in total?
   $2^{10}$, outcomes encodes as strings of $H, T$ of length 10.

   *ii*) What is the probability that exactly 3 heads comes up?
   $\frac{\binom{10}{3}}{2^{10}}$, outcomes which strings of length 10 and 3 $H$.

   *iii*) What is the probability that the outcomes give either exactly 3 heads or exactly 4 tails?
   $\frac{\binom{10}{3}+\binom{10}{4}}{2^{10}}$, outcomes which strings of length 10 and excatly 3 $H$ or exactly 4 $T$; these two cases do not overlap.