MATH 363 Discrete Mathematics
Assignment 9

Due by March 23rd

1 Grading Scheme

1. *i)-ii)* Out of **1pt**: Full marks for either of the two solutions below.
   iii)-iv) Out of **2pt**: 1pt. for correct (numerical) solution and
   1pt. for the explanation how the student obtained the result.

2. *i)* Out of **1pt**: Full marks for correct (numerical) solution.
   ii) Out of **2pt**: 1pt. for correct (numerical) solution and
   1pt. for the explanation how the student obtained the result.

3. *i)* Out of **1pt**: Full marks if they apply the conditional probability definition.
   Leave a note if the numerical solution is incorrect.
   ii) Out of **2pt**: 1pt. for each of the equations in the solution below.
   Leave a note if the numerical solution is incorrect.
   iii) Out of **2pt**: 2pt. if they apply directly Baye's formula.
   Otherwise: 1pt. if they apply the conditional probability definition and
   1pt if they use the values of parts *i)-ii)*.
   iv) Out of **2pt**: Full marks if they explain that each bit is 'sent-received' independently of the others.

4. *i)* Out of **1pt**: Full marks if they paraphrase why the two events are equal.
   ii) Out of **2pt**: Full marks if they interpret the equation: why alternating $+/-$, the indices in the sum, and the events in the probabilities on the right-hand side.
   iii) Out of **2pt**: Full marks if they paraphrase the argument below.
   iv) Out of **1pt**: Full marks if they use *iii)* and count the number of k-tuples to simplify the formula in $\hat{\text{ii}}$).

5. Out of **2pt**: Full marks if they express the random variable as a sum of random variables and paraphrase the argument below.
   1pt if they write the expected value using the definition of expected value.

6. *i)-ii)* Out of **1pt**: Full marks if either they
   write $Bin(n, p)$ and explain which are the correct parameters; or
   if they compute the probability that the random variable takes all possible values k.
   iii) Out of **3pt**: 2pt if they paraphrase at least one of the two main ideas below.
   Full marks if they paraphrase correctly the argument below.

7. *i)* Out of **1pt**: Full marks if they express coherently how to compute the probability.
   Leave a note if the numerical answer is incorrect.
   ii) Out of **2pt**: Full marks if they express coherently how to compute the expectation, or explain why we are dealing with a $Geo(p)$ and give $1/p$ as the expected value.

8. *i)* Out of **2pt**: Full marks if they express coherently how to compute the probability of $X_i = k$ or if they explain why the random variable is $Geo(p)$.

ii) Out of **1pt**: Full marks if they express coherently how to compute the expectation, or write $1/p$ as the expected value.

iii) Out of **2pt**: Full marks if they paraphrase answer below.

2 Assignment with solutions

1. Consider the following experiment. First, flip a coin. If the coin lands head, throw 1 die. If the coin lands tails, then throw two dice. Let the sample space be $S = \{h, t\} \times \{1, 2, \ldots, 12\}$.

   i) (**1pt**) Give the set $E \subset S$ that represent the event that the sum of the dice is at least 5.
   $E = \{(h, 5), (h, 6), (t, 5), (t, 6), (t, 7), (t, 8), (t, 9), (t, 10), (t, 11), (t, 12)\}$
   Alternative
   $E = \{(h, 5), (h, 6), (h, 7), (h, 8), (h, 9), (h, 10), (h, 11), (h, 12)\}$
   $\cup\{(t, 5), (t, 6), (t, 7), (t, 8), (t, 9), (t, 10), (t, 11), (t, 12)\}$,
   note that the added elements have probability zero.

   ii) (**1pt**) Give the set $F \subset S$ that represent the event that the sum of the dice is even.
   $F = \{(h, 2), (h, 4), (h, 6), (t, 2), (t, 4), (t, 6), (t, 8), (t, 10), (t, 12)\}$
   Alternative
   $F = \{(h, 2), (h, 4), (h, 6), (h, 8), (h, 10), (h, 12)\}$
   $\cup\{(t, 2), (t, 4), (t, 6), (t, 8), (t, 10), (t, 12)\}$,
   note that the added elements have probability zero.

   iii) (**2pt**) If the coin and the dice are fair. What is the probability of F
   Given that the coin lands heads, we have probability 1/2 two obtain an even number.
   Given that the coin lands tails, we also have probability 1/4 of seeing two even numbers and probability 1/4 of seeing two odd numbers (these are the two cases in which we will have an even sum). Thus

   $$P(F) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2}$$

   iv) (**2pt**) If the coin is twice as likely to land heads than land tails. What is the probability of E
   Given that the coin lands heads, we have probability 1/3 two obtain either 5 or 6.
   Given that the coin lands tails, we throw two dice. There are 36 ways in which the dice can land. Only the pair of numbers in the dice $(1, 1), (2, 2), (1, 2), (2, 1), (1, 3), (3, 1)$ give a sum less than 5. Thus

   $$P(E) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{(36 - 6)}{36} = \frac{1}{2}$$

2. *Monty Hall Three-Door contest* You have a chance to win a large prize. First, you are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host (who knows what is behind each door) opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors.

   i) (**1pt**) What is the probability that you selected the winning door, in the first place?
   1/3; there is equal chances that the prize is in the door you choose.

   ii) (**2pt**) What is the probability that win the price if you switch doors?
   2/3; there are 2 doors with no prize behind, if you pick one of them and Monty shows you another door with no prize behind; then the prize is on the third door and you win by switching. On the other hand, if you select the door with the prize, then Monty can show you either empty door and by switching you lose the prize. This is the only case out of 3 in which you lose by switching.

3. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-third of the time and a 0 two-thirds of the time, independently of earlier transmittions.

When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received incorrectly (as a 1) is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8, and the probability that it is received incorrectly (as a 0) is 0.2.

i) (**1pt**) What is the probability that the probe sent a 1 and Earth received a 1?
Let E be the event that the probe sends a one. And let R be the event that earth receives 1. Then the information we have is that $P(E) = 1/3$ and $P(R|E) = .8$. By the definition of conditional probability,

$$P(E \cap R) = P(R|E)P(E) = \frac{8}{10} \cdot \frac{1}{3} = \frac{8}{30}$$

ii) (**2pt**) What is the probability that Earth receives a 1.
We have computed the case when both the probe and earth received a one. Also, the probability that the probe sends a zero and it is received as a one is:

$$P(\bar{E} \cap R) = P(R|\bar{E})P(\bar{E}) = \frac{1}{10} \cdot \frac{2}{3} = \frac{2}{30}$$

so the total probability is

$$P(R) = P(E \cap R) + P(\bar{E} \cap R) = \frac{8}{30} + \frac{2}{30} = \frac{1}{3}.$$

iii) (**2pt**) If Earth receives a 1, what is the probability that the probe sent 1?

$$P(E|R) = \frac{P(E \cap R)}{p(R)} = \frac{8/30}{10/30} = \frac{4}{5}.$$

iv) (**2pt**) If Earth receives the string 11, what is the probability that the probe actually sent the string 11?
Since bit transmissions occur independently of each other, we want that twice:'event E occurs given that R occurs', this occurs with probability $P(E|R)^2 = \frac{16}{25}$.

4. Consider a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$; such functions are called permutations on $[n]$. If $\sigma(i) = i$ we say that i is a fixed point of σ. We will obtain the probability that a random permutation on $[n]$ has no fixed points.

   i) (**1pt**) Let D be the event that σ has no fixed points, and F_i be the event that i is a fixed point in σ. Show that $\bar{D} = \cup_{i=1}^{n} F_i$.
   The event D does not occur precisely if at least one of the points i is fixed, which is equivalent to the union of the F_i.

   ii) (**2pt**) Use the generalized inclusion-exclusion formula to show that

$$1 - P(D) = \sum_{i=1}^{n} P(F_i) - \sum_{1 \leq i_1 < i_2 \leq n} P(F_{i_1}, F_{i_k}) + \cdots \pm P(F_1, F_2, \ldots, F_n)$$

   First, by the properties of probability $P(\bar{D}) = 1 - P(D)$. Second, the generalized version of the inclusion-exclusion formula gives that, when counting the cardinality of a union, we have:

$$|\cup_{i=1}^{n} F_i| = \sum_{i=1}^{n} |F_i| - \sum_{1 \leq i_1 < i_2 \leq n} |F_{i_1} \cap F_{i_k}| + \cdots \pm \left| \bigcap_{i=1}^{n} F_i \right|;$$

   the sums are indexed so that at each term we sum over all possible collections of k of the sets F_i, with $k = 1, 2, \ldots, n$. Finally, the choice of the permutation is at random, so the probabilities are computed as favourable cases over total number of cases; dividing over $n!$ gives the formula above.

iii) (**2pt**) Explain why, for $1 \le k \le n$ and $1 \le i_1 < \cdots < i_k \le n$, then

$$P(F_{i_1}, \dots F_{i_k}) = \frac{(n-k)!}{n!}.$$

As mentioned before, $P(F_{i_1}, \dots F_{i_k}) = \frac{|\cap_{j=1}^{k} F_{i_j}|}{n!}$ so it remains to show that $| \cap_{j=1}^{k} F_{i_j}| = (n-k)!$. To see this, note that the only restriction for a permutation to satisfy the conditions $F_{i_1}, \dots, F_{i_k}$ is that the points i_j are fixed; we can order the remaining $n-k$ points as we want; there are $(n-k)!$ ways to do so.

iv) (**1pt**) Show that then

$$P(D) = \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Note that the limit of $P(D)$ is e^{-1} as $n \to \infty$.
The number of terms on each of the sums is, in fact $\binom{n}{k} = \frac{n!}{(n-k)!} k!$. And we alternate signs, this is encoded with the term $(-1)^i$. Since we can set $0! = 1$, the first term is $\frac{(-1)^0}{0!} = 1$.

5. (**2pt**) If a permutation σ on $[n]$ is chosen at random. What is the expected number of fixed points?
For this, we define a collection of random variables: let $X_i = 1$ if the event F_i occurs and zero otherwise. That means, that $X_i = 1$ if the point i is fixed and $X_i = 0$ in any other case. Note that, for all $1 \le i \le n$ we have X_i has a Bernoulli distribution with success parameter $p = \frac{1}{n}$. Thus, $E(X_i) = \frac{1}{n}$.
Finally, let X be the total number of fixed points in the random permutation. Thus, $X = X_1 + X_2 + \cdots + X_n$; by the linearity of expectations, $E[X] = n \cdot \frac{1}{n} = 1$.

6. I have 10 undistinguishable balls, I take each of them and, independently of the others, paint it red,black or pink with probability .3, .4 and .3 respectively.

i) (**1pt**) If X counts the number of red balls; what is the distribution of X?
Since the choice of colour of each ball is independent of the others, I have to perform the same experiment 10 times. If the success is to paint a ball red, then the probability of success is .3; therefore X has Binomial distribution with parameters 10 and .3.

ii) (**1pt**) If Y counts the number of black and red balls; what is the distribution of Y?
Now, the success probability is $.3 + .4 = .7$. Thus Y is distributed as a Binomial with parameters 10 and .7.

iii) (**3pt**) What is the probability that I have at least one ball of each color?
Let's define several random variables: X_r is the number of balls *painted* red, X_b is the number of balls *painted* black and X_p is the number of balls *painted* pink. These random variables have distribution Binomial with parameters $Bin(10, .3)$, $Bin(10, .4)$, $Bin(10, .3)$ respectively.
First idea; use complement: consider the complement of the event D that there is at least one ball of each color. That is, $\bar{D}$ is the event that the balls are painted with two colors or less. For example if balls are painted with black and pink then $X_r = 10$; therefore, $\bar{D} = \{X_r = 0\} \cup \{X_b = 0\} \cup \{X_p = 0\}$.
Second idea; be aware of overlaps and use inclusion-exclusion principle: Let's analyse the intersection between pairs of the events above. If $X_r = 0$ and $X_b = 0$, this means that the 10 balls are not painted red nor black; thus they are painted pink. This occurs with probability $P(X_p = 10) = .3^{10}$.
Since it is not possible that the three random variables are 0 simultaneously, we have that

$$\begin{aligned}
&P(\{X_r = 0\} \cup \{X_b = 0\} \cup \{X_p = 0\}) \\
=&P(X_r = 0) + P(X_b = 0) + P(X_p = 0) \\
&-[P(X_r = 0, X_b = 0) + P(X_b = 0, X_p = 0) + P(X_p = 0, X_r = 0)] \\
&+P(X_r = 0, X_b = 0, X_p = 0) \\
=&(.7)^{10} + (.6)^{10} + (.7)^{10} - [(.3)^{10} + (.3)^{10} + (.4)^{10}]
\end{aligned}$$

7. I collect stamps from the cereal boxes. Suppose there are 5 different stamps and that each box has a uniformly chosen stamp out of the 5 possible ones.

- (**1pt**) What is the probability that after opening 8 cereal boxes, I have at least 2 different stamps?
  We will compute the complement of this event; which is the probability that in 8 boxes that I opened, there was only one kind of stamp in all of them. This can be done in 5 different ways (each for one of the type of stamps). On the other hand, there are 5^8 possible ways to obtain 8 stamps. So the probability to have at least 2 distinct stamps is $1 - \frac{5}{5^8} = \frac{5^7 - 1}{5^7}$.
- (**2pt**) If I already have 3 different stamps. What is the expected number of boxed I will have to open before I found a stamp I don't have already?
  This expected number of boxed is modelled with a Geometric random variable, with parameter of success $p = \frac{2}{5}$. To see this, note that, once I have 3 distinct stamps, the probability that the next box contains one of the 2 stamps I don't have yet is $2/5$.
  It can be looked up, the expected value of a Geometric random variable with parameter p is $\frac{1}{p} = \frac{5}{2}$.

8. Generalize the problem above. Suppose I want to collect n stamps, and each box has a uniformly chosen stamp out of the n possible ones.

   i) (**2pt**) Given that I already have $i-1$ different stamps, let X_i be the number of boxes I have to open before I found a stamp I don't have already. What is the distribution of X_i?
   It is a geometric random variable with parameter (of success) $1 - \frac{i-1}{n}$

   ii) (**1pt**) What is the expected value of X_i?
   It is $\frac{n}{n-i+1}$

   iii) (**2pt**) What does the sum of $X_1 + X_2 + \cdots X_n$ represent?
   This sum is the number of boxes I have to open in other to complete the collection of stamps. It's expected value can be approximated as $n \ln n$ as $n \to \infty$.