Probability review

1. Definitions
2. Apply concepts

Definitions (1)

1. Probability of an event using Laplace's formula.
2. Probability function from a sample space S to $(0, 1)$.
3. Properties of a probability function.
4. Law of total probability.

Laplace's definition of Probability

- **Experiment:** procedure that **yields an outcome**.
- Sample space: the set of possible outcomes.
- Event: a subset of the sample space.
  (usually defined by a given property of the outcome)

Let S be a finite sample space of **equally likely outcomes**.
Then the probability of an event $E \subset S$ is

$$p(E) = \frac{|E|}{|S|} = \frac{\#\ \text{Favorable cases}}{\#\ \text{Total cases}}.$$

Basic properties

Probability function is $p : S \to [0, 1]$. Such that $p(S) = 1$

$$p(E) = \sum_{s \in E} p(s)$$

- Complement: $p(\bar{E}) = 1 - p(E)$
- Union: $p(E \cup F) = p(E) + p(F) - p(E \cap F)$
- Law of total probability:
  $p(E) = p(E \cap S_1) + \cdots + p(E \cap S_k)$

  when the probability space has a partition

  $$S = S_1 \cup S_2 \cup \cdots S_k.$$

Definitions (2)

1. Conditional probability of an event E given event F.
2. Bayes' formula.
3. Independent events and independent random variables.
4. Expected value of a random variable.
5. Property of linearity of expected values.

Conditional probability

Probability of E given that F occurs:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Bayes' Formula; with partition of space $S = S_1 \cup S_2 \cup \cdots S_k$:

$$\begin{aligned} p(S_1|E) = & \quad \frac{p(E|S_1)}{p(E)} \\ & = \frac{p(E|S_1)}{\sum_{i=1}^{k} p(E|S_i)p(S_i)} \end{aligned}$$

Random variables

Random variables come from experiments
where the outcomes are numbers, say natural numbers.

Probability distribution $=$
Probability function of a distinctive type.

Independence

Independent E, F events if:

$$p(E \cap F) = p(E)p(F)$$

Independent random variables X and Y if:
For all $k, l \in \mathbb{R}$,

$$p(X = k, Y = l) = p(X = k)p(Y = l)$$

Expectation and linearity

Expected value of a random variable:

$$\mathbb{E}(X) = \sum_{k \in \mathbb{Z}} k \cdot p(k)$$

It is a (deterministic) estimated value of the random outcome.

Linearity: If X can be written as: $X = X_1 + \cdots + X_n$

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n)$$

E.g. Counting the number of heads in n coin flips.

Distribution vs. Expectation

Probability distribution:

It describes the probability of observing each possible random outcome.

Expected value of a random variable:

It is a (deterministic) estimated value of the random outcome.

Apply concepts (General)

1. Given an experiment and an event E. Define the sample space S, the probability function and compute the probability of event E.
   - If the outcomes are equally likely, use Laplace's formula.
2. Given a random variable X, compute its expected value.
   - Use the definition or,
   - Write X as a sum and use linearity.

Apply concepts (In detail)

Given a problem involving probability,

- Recognize independent events.
- Recognize random variables as sum of simpler r.v.'s
- Compute probability of a union: inclusion-exclusion
- Compute probability according to cases: law of total probability
- Compute conditional probabilities: Bayes' theorem