

# Announcements

Monday, August 28

- ▶ Piazza polls will start counting for participation today.
  - ▶ Ask your neighbor If you couldn't vote on Friday's poll.
- ▶ Homework this week is due *Friday 11:59pm*.
- ▶ Subsequent homeworks will be due on Wednesdays.
- ▶ This term, the quiz' length will be 10 min long.
- ▶ This week it will cover any material from August 23rd and 28th.
- ▶ A missing link from last lecture:

[two planes intersecting]

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

This is the kind of problem we'll talk about for the first half of the course.

- ▶ A **solution** is a list of numbers  $x, y, z, \dots$  that make *all* of the equations true.
- ▶ The **solution set** is the collection of all solutions.
- ▶ **Solving** the system means finding the solution set.

What is a *systematic* way to solve a system of equations?

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

What strategies do you know?

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

**Elimination method:** in what ways can you manipulate the equations?

# Solving Systems of Equations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Multiply first by  $-3$

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Add first to third

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$-5y - 10z = -20$$

Now I've eliminated  $x$  from the last equation!

...but there's a long way to go still. Can we **make our lives easier**?

# Solving Systems of Equations

Better notation

It sure is a pain to have to write  $x, y, z$ , and  $=$  over and over again.

**Matrix notation:** write just the numbers, in a box, instead!

$$\begin{array}{rcl} x + 2y + 3z & = & 6 \\ 2x - 3y + 2z & = & 14 \\ 3x + y - z & = & -2 \end{array} \quad \begin{array}{c} \text{becomes} \\ \text{~~~~~} \end{array} \quad \left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ▶ **Multiply** all entries in a row by a nonzero number. (scale)
- ▶ **Add a multiple** of each entry of one row to the corresponding entry in another. (row replacement)
- ▶ **Swap** two rows. (swap)

[interactive row reducer]

# Row Operations

## Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Start:

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

**Goal:** we want our elimination method to eventually produce a system of equations like

$$x = a$$

$$y = b \quad \text{or in matrix form,}$$

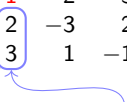
$$z = c$$

So we need to do row operations that make the start matrix look like the end one.

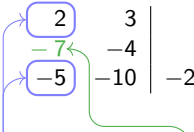
**Strategy:** fiddle with it so we only have ones and zeros. [\[animated\]](#)

# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$


We want these to be zero.  
So we subtract multiples of the first row.

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$


We want these to be zero.

It would be nice if this were a 1.  
We could divide by  $-7$ , but that  
would produce ugly fractions.

Let's swap the last two rows first.



# Row Operations

Continued

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

We want these to be zero.

Let's make this a 1 first.

Success!

Check:

$$\begin{array}{rcl} x + 2y + 3z & = & 6 \\ 2x - 3y + 2z & = & 14 \\ 3x + y - z & = & -2 \end{array}$$

substitute solution  
~~~~~>

# Row Equivalence

## Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

## Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the **linear equations of row-equivalent matrices** have the *same solution set*.

# A Bad Example

## Example

Solve the system of equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

Let's try doing row operations:

First clear these by  
subtracting multiples  
of the first row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right)$$

Now clear this by  
subtracting  
the second row.

$$\left( \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

## Continued

translates into

$$\begin{array}{rcl} x + y = 2 & & x + y = 2 \\ 3x + 4y = 5 & \text{have the same solutions as} & y = -1 \\ 4x + 5y = 9 & & 0 = 2 \end{array}$$

### Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

# Section 1.2

## Row Reduction and Echelon Forms

# Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a “solved” matrix. What do we mean by “solved”?

A matrix is in **row echelon form** if

1. All *zero rows* are at the bottom.
2. Each *leading nonzero entry* of a row is to the *right* of the leading entry of the row above.
3. *Below a leading entry* of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$  = any number

$\boxed{\star}$  = any nonzero number

## Definition

A **pivot**  $\boxed{\star}$  is the *first nonzero entry of a row* of a matrix in row echelon form.

# Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The *pivot* in each nonzero row is *equal to 1*.
5. Each pivot is the *only nonzero entry* in its column.

Picture:

$$\begin{pmatrix} \color{red}{1} & 0 & \star & 0 & \star \\ 0 & \color{red}{1} & \star & 0 & \star \\ 0 & 0 & 0 & \color{red}{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} \star = \text{any number} \\ \color{red}{1} = \text{pivot} \end{array}$$

**Note:** Echelon forms do not care whether or not a column is augmented. Just *ignore the vertical line*.

## Question

Can every matrix be put into reduced row echelon form only using row operations?

**Answer:** Yes! We'll see this shortly.

# Reduced Row Echelon Form

Continued

Why is this the “solved” version of the matrix?

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

is in reduced row echelon form. It *translates into*

which is clearly *the solution*.

But what happens if there are *fewer pivots* than variables? ... *parametrized* solution set (later).





# Reduced Row Echelon Form

## Theorem

Every matrix is *row equivalent to* one and *only* one matrix in *reduced row echelon form*.

We'll give an algorithm, called **row reduction**, which demonstrates that every matrix is *row equivalent to at least one* matrix in reduced row echelon form.

**Note:** Like echelon forms, the row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

The uniqueness statement is interesting—it means that, *nomatter how* you row reduce, you *always get the same matrix* in reduced row echelon form. (Assuming you only do the three legal row operations... and you **don't make any arithmetic errors**.)

Maybe you can figure out why it's true!