Announcements

Wednesday, August 30

- Last change to advise of any conflict between exam dates and religious holidays.
- Quiz tip: Always check your answers,

  substitute solution into initial system of equations

$$
\begin{aligned}
1 + 2\cdot(-2) + 3\cdot 3 &= 6 \\
2\cdot 1 - 3\cdot(-2) + 2\cdot 3 &= 14 \qquad \checkmark \\
3\cdot 1 + (-2) - 3 &= -2
\end{aligned}
$$

Row Equivalence

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

Inconsistent system

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Using row operations: [Interactive row reducer]

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow{\text{is row equivalent to}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

In the latter system, there is no way to make all equations true, so our **original system has no solutions** either.

Section 1.2

Row Reduction and Echelon Forms

Row Echelon Form

We'll come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. But *first*, what do we mean by "solved"?

> A matrix is in **row echelon form** if
>
> 1. All *zero rows* are at the bottom.
> 2. Each *leading nonzero entry* of a row is to the *right* of the leading entry of the row above.
> 3. *Below a leading entry* of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number

$\boxed{\star}$ = any nonzero number

Definition

A **pivot** $\boxed{\star}$ is the *first nonzero entry of a row* of a matrix in row echelon form.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The *pivot* in each nonzero row is *equal to* 1.
5. Each pivot is the *only nonzero entry* in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star =$ any number

$1 =$ pivot

Note: Echelon forms do not care whether or not a column is augmented. Just *ignore the vertical line*.

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! We'll see this at the end of class.

Reduced Row Echelon Form

Continued

A reduced echelon form is the "solved" version of its original augment matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Because it clearly *translates into the solution*

There are **two caveats** here:

1. What if the *last column* has a *pivot*? . . . *inconsistent* system of equations.
2. What if there are *fewer pivots* than variables? . . . *parametrized* solution set.

Inconsistent Matrices

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & \color{red}{0} \\ 0 & 1 & \star & \star & \color{red}{0} \\ 0 & 0 & 0 & 0 & \color{red}{1} \end{array}\right)$$

$\star$ = any number

$\color{red}{1}$ = pivot

A reduce echelon form corresponds to an **inconsistent system of equations** if and only if *the last* (i.e., the augmented) *column is a pivot column.*

Example

$$\left(\begin{array}{cccc|c} \color{red}{1} & 0 & 2 & 0 & 3 \\ 0 & \color{red}{1} & 2 & 0 & 1 \\ 0 & 0 & 0 & \color{red}{1} & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{translates into}}$$

Free Variables

Consider a row echelon form A which has *no pivot on the last column*, assume that it corresponds to a linear system of equations in the variables $x_1, \ldots, x_n$.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

We say that x_i is a **free variable** if its corresponding column in A is *not a pivot column*. (What about the last column?)
(Last column doesn't correspond to a variable.)

Example

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|r} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right)$$

The free variables are x_2 and x_4: their columns are *not pivot columns*.

Free variables

Continued

The solution set is obtained by moving all free variables to the right-hand side of the $=$.

The form translates into the system of equations

$$\begin{cases} x_1 \qquad\qquad + 3x_4 = 2 \\ \qquad\qquad x_3 + 4x_4 = -1 \end{cases} \implies \boxed{\begin{aligned} x_1 &= 2 - 3x_4 \\ x_3 &= -1 - 4x_4 \end{aligned}}$$

The **solution in parametric form** is, for any values of x_2 and x_4,

Important

In a (consistent) linear system, you can choose *any value* for the free variables!

Poll

Poll

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Reduced Row Echelon Form

Theorem

Every matrix is *row equivalent to* one and *only* one matrix in *reduced row echelon form*.

Maybe you can figure out why it's true!

The uniqueness statement is interesting—it means that, *nomatter how* you row reduce, you *always get the same matrix* in reduced row echelon form. (Assuming you only do the three legal row operations... and you don't make any arithmetic errors.)

Important

We'll give an algorithm, called **row reduction**, which demonstrates that every matrix is *row equivalent to at least one* matrix in reduced row echelon form.

Note: The row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

Row Reduction Algorithm

Step 1a	Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).
Step 1b	Scale 1st row so that its leading entry is equal to 1.
Step 1c	Use row replacement so all entries below this 1 are 0.
Step 2a	Swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row.
Step 2b	Scale 2nd row so that its leading entry is equal to 1.
Step 2c	Use row replacement so all entries below this 1 are 0.
Step 3a	Swap the 3rd row with a lower one so that the leftmost nonzero (uncovered) entry is in 3rd row.
etc.	
Last Step	Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Example (Example 1)

$$\left(\begin{array}{ccc|c} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

[animated]

Row Reduction

Example 1

$$\left(\begin{array}{rrr|r} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Row Reduction

Example 1, continued

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

$$\begin{array}{r} \\ \text{“Active” row} \longrightarrow \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right)$$

Row Reduction

Example 1, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Row Reduction

Example 1, continued

Success! The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \Longrightarrow \qquad \left\{\begin{array}{rrrrr} x & & & = & 1 \\ & y & & = & -2 \\ & & z & = & 3 \end{array}\right.$$

Row Reduction

Example 2 (inconsistent system)

The linear system

$$\begin{aligned} 2x + 10y &= -1 \\ 3x + 15y &= 2 \end{aligned}$$

gives rise to the matrix

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array}\right)$$

The row reduced matrix

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

corresponds to the *inconsistent* system

$$\begin{aligned} x + 5y &= 0 \\ 0 &= 1. \end{aligned}$$

Row reduction

Example 3 (Parametrized solution)

The linear system

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$

gives rise to the matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

The row reduced matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$

corresponds to the linear system

$$\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$$

Row reduction

Example 3 (Parametrized solution), continued

The system

$$\begin{aligned} x + 5z &= 1 \\ y + 2z &= -1 \end{aligned}$$

comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \end{aligned}$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

So we have found the solution set: it is all values x, y, z where

$$\begin{aligned} x &= 1 - 5z \\ y &= -1 - 2z \qquad \text{for } z \text{ any real number.} \\ (z &= z) \end{aligned}$$

This is called the **parametric form** for the solution. [interactive picture]