When a vector is a linear combinations of...

Review

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{aligned} x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3 \end{aligned} \quad \xrightarrow{\text{matrix form}} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{row reduce}} \quad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{solution}} \quad \begin{aligned} x_1 &= -1 \\ x_2 &= -9 \end{aligned}$$

Conclusion:
Yes! Use the weights $x_1 = -1$ and $x_2 = -9$:

$$(-1)\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + (-9)\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Poll 1, also answer the matrix poll

Section 1.4

The Matrix Equation $Ax = b$

Today: Spans and Solutions to Equations

Let $\mathbf{b} \in \mathbf{R}^n$ and A be a matrix with columns $v_1, v_2, \ldots, v_n \in \mathbf{R}^n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Very Important Fact

$Ax = b$ has a solution

$\iff$ there exist $x_1, \ldots, x_n$ such that $A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$

"if and only if"

$\iff$ there exist $x_1, \ldots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$

$\iff$ b is a linear combination of $v_1, \ldots, v_n$

$\iff$ b is in the span of the columns of A.

Matrix $\times$ Vector

the first number is
the number of rows

the second number is
the number of columns

Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

this means the equality
is a *definition*

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \boxed{\overset{\text{def}}{=}} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

these must be equal

The output is a vector in $\mathbf{R}^m$.

Necessary: Number of columns of A equals number of rows of x.

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Definition

A *row vector* is a matrix with one row. The **product** of a row vector of length n and a (column) vector of length n **is a scalar!**

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} a_1x_1 + \cdots + a_nx_n.$$

If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

$$Ax = \begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1x \\ r_2x \\ \vdots \\ r_mx \end{pmatrix}$$

This is **a vector in $\mathbf{R}^m$**.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

Note this is the same as before:

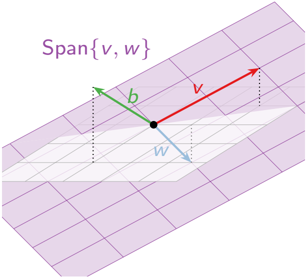
Now you have **two ways of computing** Ax.

In the *second*, you calculate Ax *one entry at a time*.

Both are convenient, so we'll use both.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Output vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A?

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

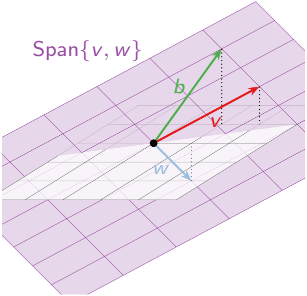
In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A?

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

This is consistent with the picture on the previous slide:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Linear Systems, Vector Equations, Matrix Equations, . . .

Now have four equivalent ways of writing linear systems:

1. As a *system of equations*:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an *augmented matrix*:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a *vector equation* ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a *matrix equation* ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again.

When Solutions Always Exist

Here are **criteria** for a linear system **to always have a solution**.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are **equivalent**

1. $Ax = b$ has a *solution for all b* in $\mathbf{R}^m$.
2. The span of the columns of A is *all of $\mathbf{R}^m$*.
3. A has a pivot *in each row*.

recall that this means that for given A, either they're all true, or they're all false

Why is (1) the same as (3)?

Look at **reduced echelon** forms of A.

- If A has *a pivot in each row*:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\,A \mid b\,) \text{ reduces to:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

  There's no b that makes it inconsistent, so there's *always a solution*.

- If A *doesn't have a pivot* in each row:

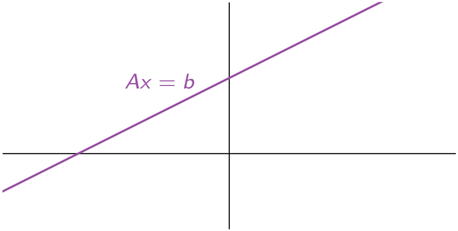
$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be made inconsistent:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$

Section 1.5

Solution Sets of Linear Systems

Plan For Today

Describe and draw the solution set of $Ax = b$, using spans.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Example 1

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (*everything to the right of the* $=$) was zero to begin, it will always *stay zero!*

Example 2

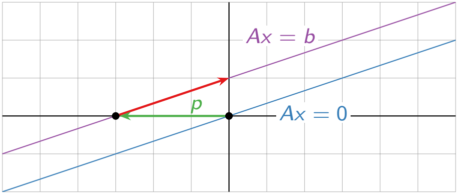
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Example 2, explained

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equation}} \quad x_1 - 3x_2 = -3$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Note that p is itself a solution: take $x_2 = 0$.

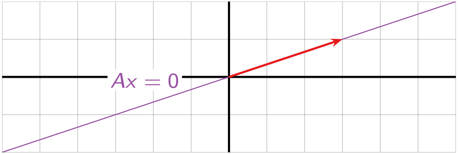
Example 3

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Example 3, explained

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 &= 3x_2 \\ x_2 &= x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

These equations are called the **parametric vector form** of the solutions.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Parametric Vector Form and Span

In general

Let A be an $m \times n$ matrix. If the *free variables* in the equation $Ax = b$ are $x_i, x_j, x_k, \ldots$

And the **parametric vector form** of the solution is

$$x = b' + x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $b', v_i, v_j, v_k, \ldots$ in $\mathbf{R}^n$, and any scalars $x_i, x_j, x_k, \ldots$

Then the solution set is

$$b' \quad + \quad \mathrm{Span}\{v_i,\ v_j,\ v_k, \ldots\}.$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\quad \text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \left\{ \begin{aligned} x_1 \qquad - 8x_3 - 7x_4 &= 0 \\ x_2 + 4x_3 + 3x_4 &= 0 \end{aligned} \right.$$

$$\xrightarrow{\text{parametric form}} \left\{ \begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \right.$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Everything is easier when $b = 0$, so we start with this case.

Definition

A system of linear equations of the form $Ax = 0$ is called *homogeneous.*

A homogeneous system *always has the solution* $x = 0$. This is called the *trivial solution.* The nonzero solutions are called **nontrivial.**

Observation

$Ax = 0$ has a *nontrivial* solution

$\iff$ there is a free variable

$\iff$ A has a *column with no pivot*.

The opposite:

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **nonhomogeneous** or **inhomogeneous.**

Poll

Solutions for Homogeneous Systems

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u+v) = Au + Av$
- $A(cv) = cAv$

See Lay, §1.4, Theorem 5.

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in $\mathrm{Span}\{u, v\}$. Why?

Important

The set of **solutions to $Ax = 0$** is **a span**.

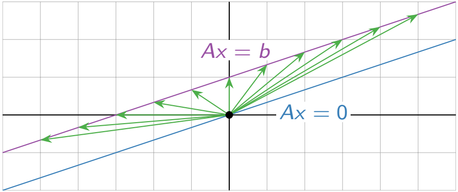
Solutions for **Consistent** Nonhomogeneous Systems

When consistent

The set of **solutions to $Ax = b$**, is **parallel** to a span.

Why? solutions are obtained by taking one **specific** or **particular solution** p to $Ax = b$, and *adding all solutions to $Ax = 0$*.

If $Ap = b$ and $Ax = 0$, then



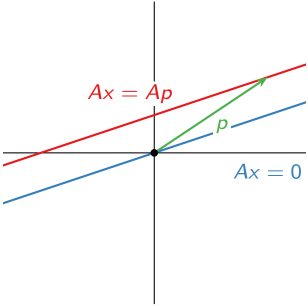
Note:

Works for *any specific solution p*: it doesn't matter how one found it!

Reverse Engineering: Nonhomogeneous System

Question

Give a system whose solution set passes through point p and it is *parallel to* the solution set of $Ax = 0$.



1. Set $b = Ap$.
   Entries in p are the weights that produce b *as a linear combination* of columns of A.
2. Now *p is a specific solution* to $Ax = b$,
3. so $Ax = b$ **is the system** we wanted.

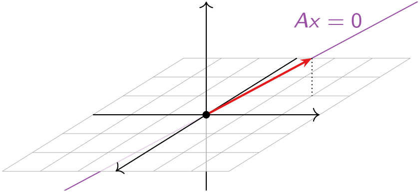
Take out: If we describe the solution set of $Ax = 0$,
then we can describe the solution set of $Ax = b$ for *all b* in the *Span of columns of A*.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

Answer: Span $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = 0 \\ \phantom{x_1 - {}} x_2 + x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

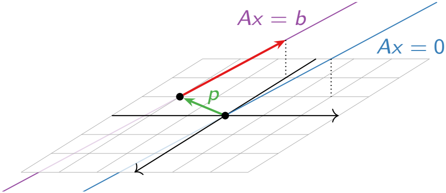
$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

Answer: $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$



The solution set is a *translate* of

$$\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}:$$

it is the parallel line through

$$p = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & -5 \\ 2 & -1 & -5 & -3 \\ 1 & 0 & -2 & -2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 - 2x_3 = -2 \\ x_2 + x_3 = -1 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 - 2 \\ x_2 = -x_3 - 1 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$