Announcements

Wednesday, September 20

- Quiz 3: Come forward to pick up your exam

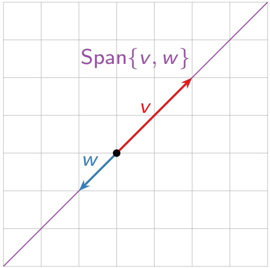
- First time I was away of home: Masters in Montreal
  - Life on campus was too expensive for me
  - I couldn't find people that I felt comfortable with (cultural clash)
  - School was ok, though I only took two courses
  - I didn't know how to ask my family for more attention
- Don't hesitate to **use the resources** on campus

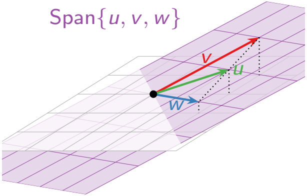
Section 1.7

Linear Independence

Motivation

Sometimes the *span* of a set of vectors *"is smaller"* than you expect from the number of vectors.





This "means" you *don't need so many vectors* to express the same set of vectors.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

has **only the trivial solution** $x_1 = x_2 = \cdots = x_p = 0$.

The opposite:

The set $\{v_1, v_2, \ldots, v_p\}$ is *linearly dependent* if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0.$$

This is called a *linear dependence relation*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

The notion of linear (in)dependence *applies to a collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Linear Independence and Matrix Columns

By definition, $\{v_1, v_2, \ldots, v_p\}$ is *linearly independent* if and only if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$$

has only the trivial solution. This holds *if and only if* the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the *matrix with columns $v_1, v_2, \ldots, v_p$*:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has *a pivot in each column*.

If one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones:

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

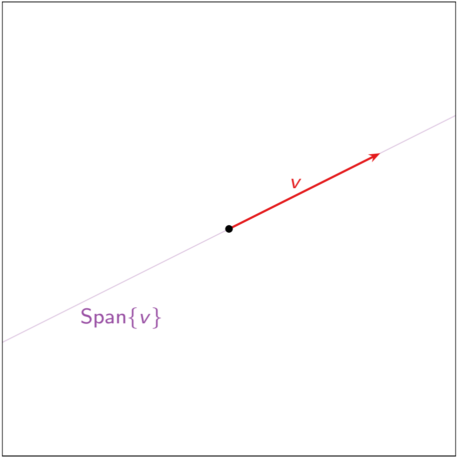
Then the vectors are linearly *de*pendent:

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0,$$

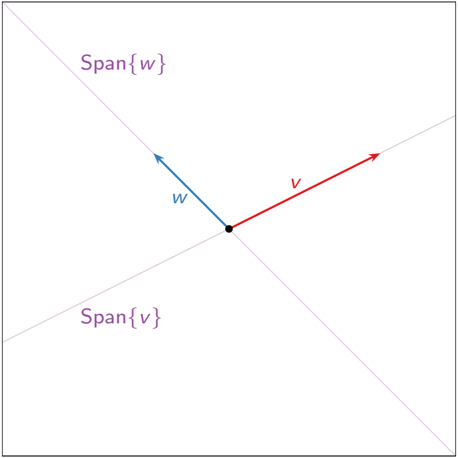
Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** if and only if *one* of the vectors is *in the span of the other* ones.



In this picture

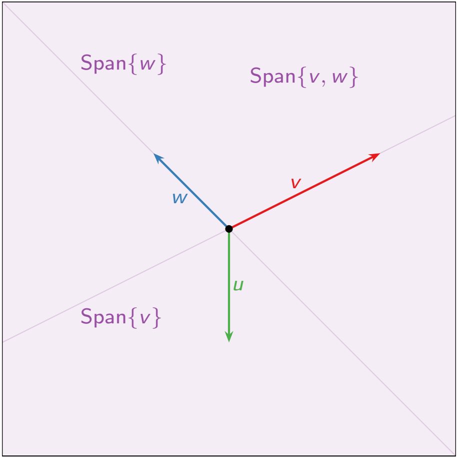
One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.



In this picture

One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.



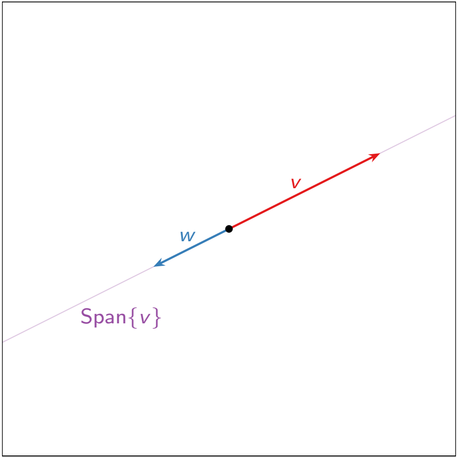
In this picture

One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

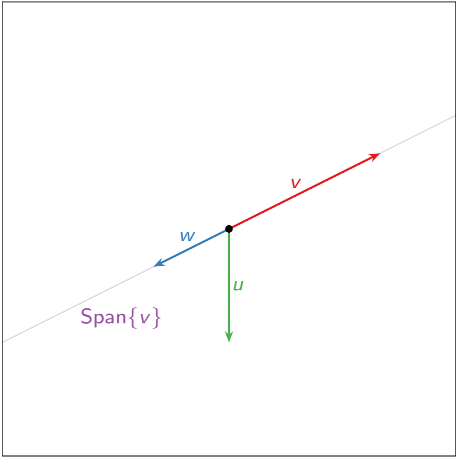
Three vectors $\{v, w, u\}$:
Linearly dependent: u is in $\mathrm{Span}\{v, w\}$.

Also
v is in $\mathrm{Span}\{u, w\}$ and
w is in $\mathrm{Span}\{u, v\}$.



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

- *Two vectors* are linearly **dependent** if and only if they are *collinear*.



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in $\mathrm{Span}\{v\}$ (and vice-versa).

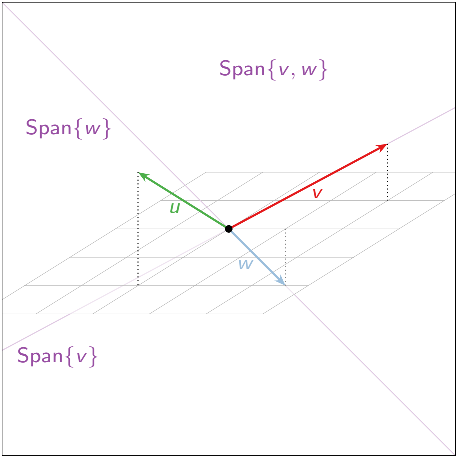
- *Two vectors* are linearly **dependent** if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: w is in $\mathrm{Span}\{v\}$ (and vice-versa).

- If a *set of vectors* is linearly **dependent**, then so is *any larger set* of vectors!

Linear Independence

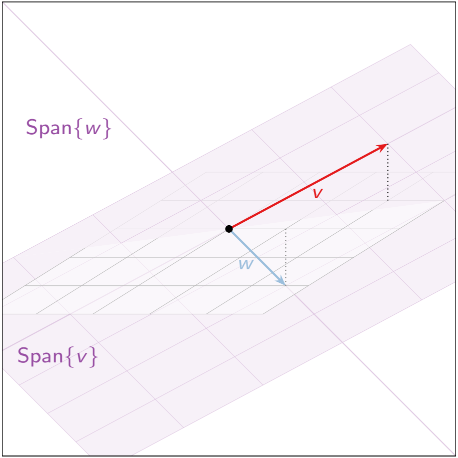
Pictures in $\mathbf{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

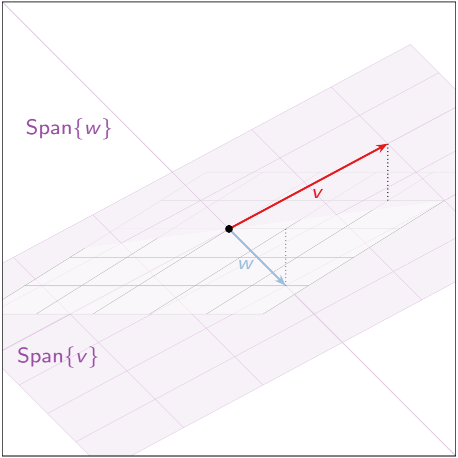
Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in $\mathrm{Span}\{v, w\}$.



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in Span$\{v, w\}$.

Which subsets are linearly dependent?

Suppose a set of vectors $\{v_1, v_2, \dots, v_p\}$ is *linearly dependent*.

Take the **largest** j such that v_j *is in the span* of the others.

Is v_j is in the *span of* $v_1, v_2, \dots, v_{j-1}$?

For example, $j = 3$ and

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

Better Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** if and only if there is some j such that v_j *is in* $\mathrm{Span}\{v_1, v_2, \dots, v_{j-1}\}$.

If the vector v_j **is not in** $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$,

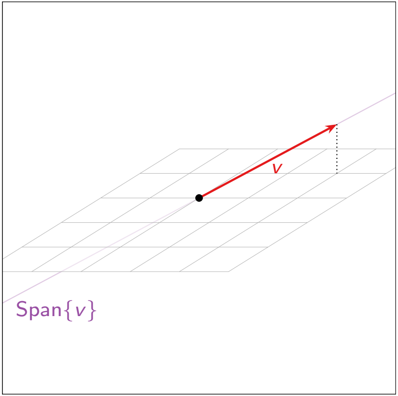
it means $\text{Span}\{v_1, v_2, \ldots, v_j\}$ *is bigger* than $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

If true for all j

A set of vectors is linearly independent if and only if, every time *you add another vector* to the set, the *span gets bigger*.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

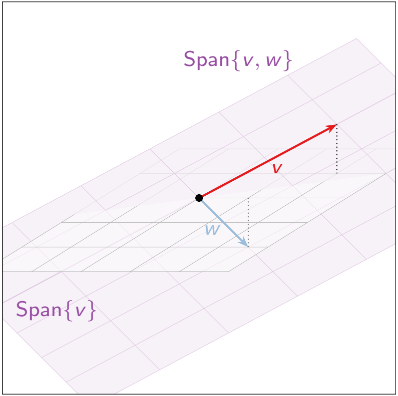


One vector $\{v\}$:

Linearly independent: span got bigger (than $\{0\}$).

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every* j, the *span of* $v_1, v_2, \ldots, v_j$ *is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

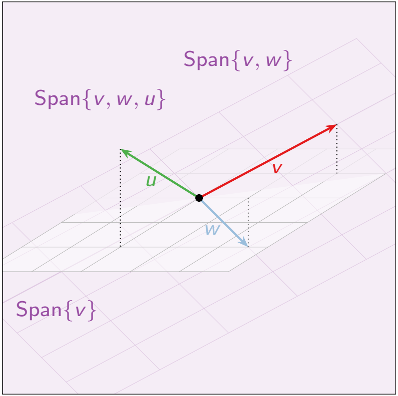


One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.



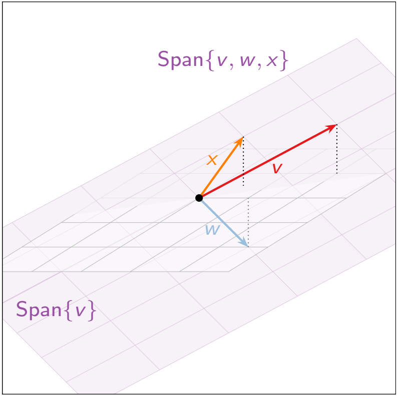
One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.



One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, x\}$:
Linearly dependent: span didn't get bigger.

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. *If $n > m$* then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*:

A wide matrix can't have linearly independent columns.

Fact 2: *If one* of $v_1, v_2, \ldots, v_n$ *is zero*, then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*.

A set containing the zero vector is linearly dependent.

Section 1.8

Introduction to Linear Transformations

Motivation

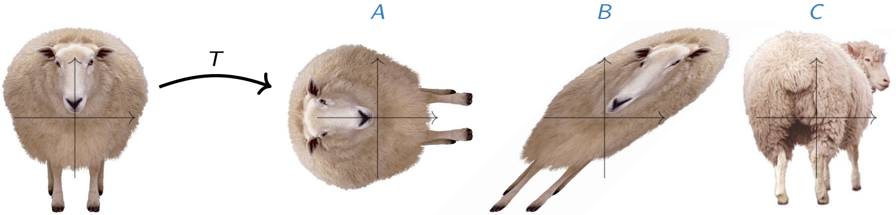
Let A be an $m \times n$ matrix. For $Ax = b$ we can describe

- the **solution set**: all x **in** $\mathbf{R}^n$ making the **equation true**.

- the *column span*: the set of all b *in* $\mathbf{R}^m$ making the *equation consistent*.

It turns out these two sets are very closely related to each other.

Geometry matrices: *linear transformation* from $\mathbf{R}^n$ to $\mathbf{R}^m$.



Transformations

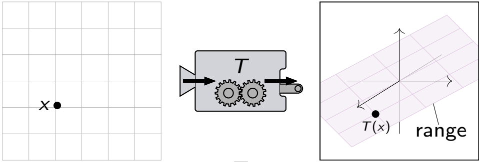
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x *in* $\mathbf{R}^n$ a vector $T(x)$ *in* $\mathbf{R}^m$.

- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{\, T(x) \mid x \text{ in } \mathbf{R}^n \}$ is the **range** of T.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$



$$\underset{\text{domain}}{\mathbf{R}^n} \xrightarrow{\quad\quad T \quad\quad} \underset{\text{codomain}}{\mathbf{R}^m}$$

Think of T as a *"machine"*

- takes x as an input
- *gives you* $T(x)$ as the output.

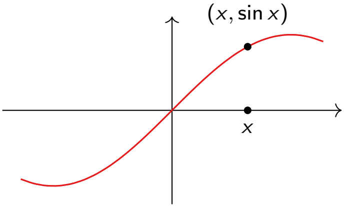
Functions from Calculus

Many of the functions you know have domain and codomain $\mathbf{R}$.

For example, $f : \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$

Often times *we omit the name $f(x)$ of the function* "x^2".

You may be used to thinking of a function in terms of its graph. E.g.,



The horizontal axis is the *domain*, and the vertical axis is the *codomain*.

This is fine when the domain and codomain are $\mathbf{R}$, but **it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$**!

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

In other words, *T takes* the vector *x in $\mathbf{R}^n$ to* the vector *Ax in $\mathbf{R}^m$*.

- The domain of T is $\mathbf{R}^n$, which is the number of columns of A.
- The codomain of T is $\mathbf{R}^m$, which is the number of rows of A.
- The **range** of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1v_1 + x_2v_2 + \cdots + x_nv_n.$$

This is the **column span of A**. It is *a span of vectors in the codomain*.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) =$
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

Matrix Transformations

Example, continued

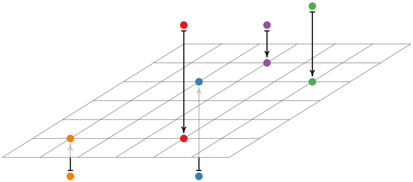
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?
- Find c such that there is *no* v with $T(v) = c$.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^3 \to \mathbf{R}^3$. Then

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

This is *projection onto the xy-axis*. Picture:



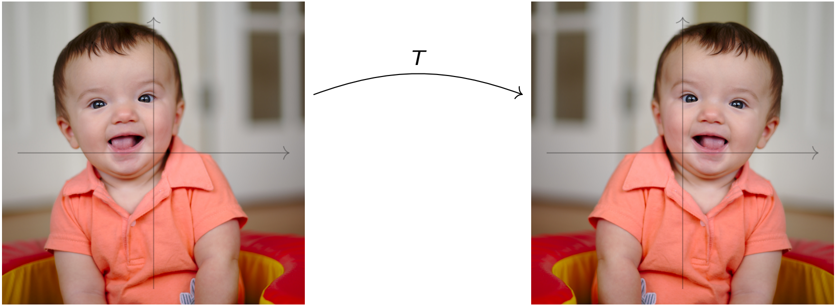
Matrix Transformations

Reflection

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

This is *reflection over the y-axis*. Picture:



Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \qquad T(cv) = cT(v).$$

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for *all vectors* u, v in $\mathbf{R}^n$ and *all scalars* c.

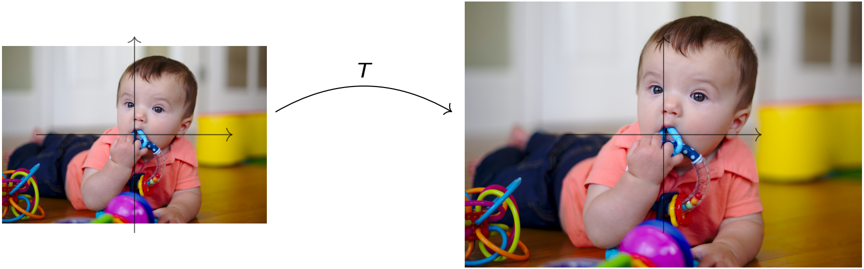
In other words, T **"respects" addition and scalar multiplication**.

More generally, (in engineering this is called **superposition**)

$$T\big(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\big) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear?

This is called **dilation** or **scaling** (by a factor of 1.5). Picture:



Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. Is T linear?

This is called **rotation** (by 90°). Picture:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$
$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$
$$T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$