Announcements

Monday, September 25

- Webwork is due by Friday

Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{only when} \quad a_1 = a_2 = \cdots = a_n = 0.$$

Otherwise they are *linearly dependent*, and an a 'witness' equation $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ with *some* $a_i \neq 0$ is a linear dependence relation.

Theorem

Let A be the $m \times n$ matrix with column vectors $v_1, v_2, \ldots, v_n$ in $\mathbf{R}^m$.
If the *vectors are linearly dependent*, a *nontrivial solution* to the matrix equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \qquad \begin{array}{c} \text{gives a linear} \\ \text{dependence relation} \end{array} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0.$$

The following *are equivalent*:

1. The set $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent**.
2. $Ax = 0$ only has the trivial solution.
3. A has a pivot in every column
4. For every i between 1 and n, v_i is not in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\}$.

Section 1.8

Introduction to Linear Transformations

Motivation

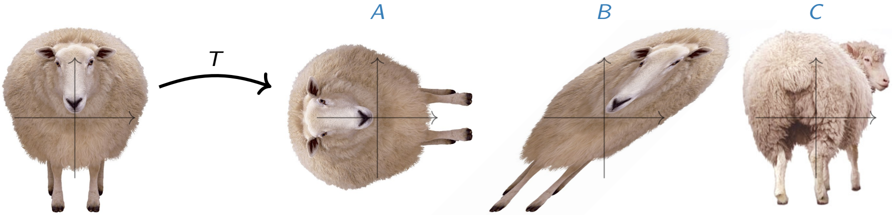
Let A be an $m \times n$ matrix. For $Ax = b$ we can describe

- the **solution set**: all x **in** $\mathbf{R}^n$ making the **equation true**.

- the *column span*: the set of all b *in* $\mathbf{R}^m$ making the *equation consistent*.

It turns out these two sets are very closely related to each other.

Geometry matrices: *linear transformation* from $\mathbf{R}^n$ to $\mathbf{R}^m$.

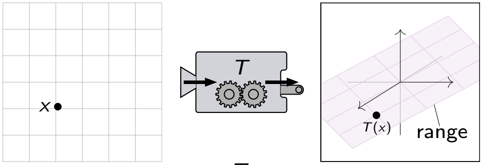


Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector *x in $\mathbf{R}^n$* a vector *$T(x)$ in $\mathbf{R}^m$*.

Notation:

$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.



$$\mathbf{R}^n \xrightarrow{\quad T \quad} \mathbf{R}^m$$

domain — codomain

Think of T as a *"machine"*

- takes x as an input
- *gives you $T(x)$* as the output.

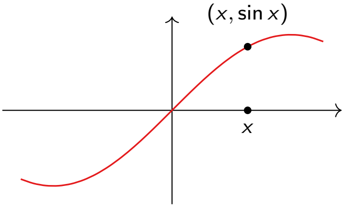
Functions from Calculus

Many of the functions you know have domain and codomain $\mathbf{R}$.

$$\text{For example, } f : \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Often times *we omit the name $f(x)$ of the function* "x^2".

You may be used to thinking of a function in terms of its graph. E.g.,



The horizontal axis is the *domain*, and the vertical axis is the *codomain*.

This is fine when the domain and codomain are $\mathbf{R}$, but **it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$**!

Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

In other words, *T takes* the vector *x in $\mathbf{R}^n$ to* the vector *Ax in $\mathbf{R}^m$*.

- The domain of T is $\mathbf{R}^n$, which is the number of columns of A.
- The codomain of T is $\mathbf{R}^m$, which is the number of rows of A.
- The **range** of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the **column span of A**. It is *a span of vectors in the codomain*.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) =$
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

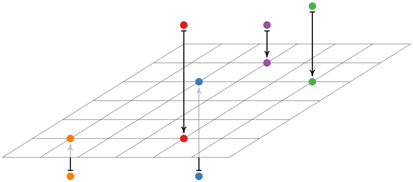
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?
- Find c such that there is *no* v with $T(v) = c$.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^3 \to \mathbf{R}^3$. Then

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

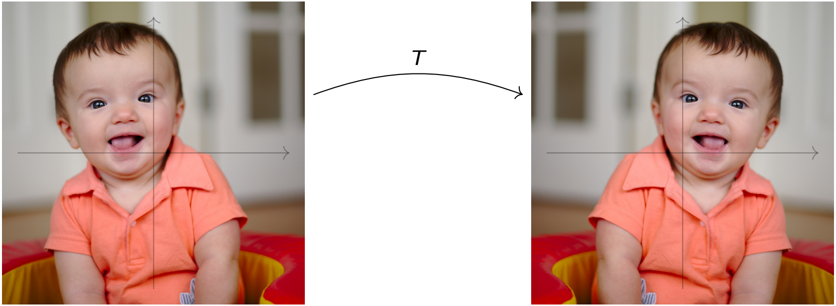
This is *projection onto the xy-axis*. Picture:



Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

This is *reflection over the y-axis*. Picture:



Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for *all vectors* u, v in $\mathbf{R}^n$ and *all scalars* c.

In other words, T **"respects" addition and scalar multiplication**.

More generally, (in engineering this is called **superposition**)

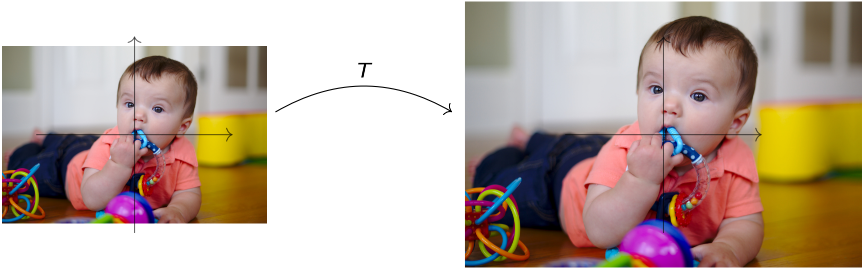
$$T\big(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n\big) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$

Linear Transformations

Dilation

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear?

This is called **dilation** or **scaling** (by a factor of 1.5). Picture:



Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. Is T linear?

This is called **rotation** (by 90°). Picture:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$