- Webwork is due by Friday
- No quiz for this week
- Please fill out a feedback form in Room: ESLAVA at socrative.com
 - It is anonymous:(it asks for a name, but you can make one up)
 - Feedback form will remain open until Friday noon.
- Midterm grades
 - Grades will be entered will be entered by Friday noon
 - Handed back during recitation
 - Notify your TA of any concern
 - Verify grade in T-square matches your hardcopy
- Progress report: To help you tune up studying strategies
 - Satisfactory if current grade is at least 70%
 - 75% Midterm
 - 15% Quizzes (no drops)
 - 5% Webwork (no drops)
 - 5% Participation
- No grade discussion by email

Section 1.9

The Matrix of a Linear Transformation

Unit Coordinate Vectors



Important: if A is an $m \times n$ matrix with columns v_1, v_2, \ldots, v_n , then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: the transformation T(x) = Ax sends e_i to vector v_i .

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}$$

Recap: Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then $A(u+v) = Au + Av \qquad A(cv) = cAv.$ So if T(x) = Ax is a matrix transformation then, $T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$

Definition

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if it satisfies the above equations for all vectors u, v in \mathbb{R}^n and all scalars c.

In other words, *T* "respects" addition and scalar multiplication.

More generally, (in engineering this is called **superposition**)

$$T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

So that *unit coordinate vectors* determine where all vectors in \mathbf{R}^n get mapped to in \mathbf{R}^m .

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: T(x) = Ax. The matrix A is called the *standard matrix for* T.



Dictionary

Linear transformation $m \times n \text{ matrix } A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & | \end{pmatrix}$ T(x) = Ax

Linear Transformations: Dilation

Before, we defined a **dilation** transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ by T(x) = 1.5x. What is its standard matrix?

$$T(e_1) = 1.5e_1 = \begin{pmatrix} 1.5\\0 \end{pmatrix}$$
$$T(e_2) = 1.5e_2 = \begin{pmatrix} 0\\1.5 \end{pmatrix}$$
$$\implies A = \begin{pmatrix} 1.5&0\\0&1.5 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}.$$

Construction Phase 1

Question



Construction Phase 2

Question



Construction Phase 3

Question



Resulting matrix

Question

$$T(e_{1}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$T(e_{1}) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Linear Transformations: Rotation

Question

What is the matrix for the linear transformation $\mathcal{T}\colon \mathbf{R}^2 o \mathbf{R}^2$ defined by

T(x) = x rotated counterclockwise by an angle θ ?



There is a long list of geometric transformations of R^2 in $\S1.9$ of Lay. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, \ldots) Please look them over.

Onto Transformations

Definition

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is **onto** (or **surjective**) if the range of T is equal to \mathbb{R}^m (its codomain). In other words, each b in \mathbb{R}^m is the image of at least one x in \mathbb{R}^n : every possible output has an input. Note that not onto means there is some b in \mathbb{R}^m which is not the image of any x in \mathbb{R}^n .



Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- ► T is onto
- T(x) = b has a solution for every b in \mathbf{R}^m
- Ax = b is consistent for every b in \mathbf{R}^m
- A has a pivot in every row
- **•** The columns of A span **R**^m

Question

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m? Answer: A must have *at least as many columns as rows* $(m \le n)$ to have a pivot in every row.

 $\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$ For instance, **R**² is "too small" to map *onto* **R**³.

 Poll

 If $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto,

 what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A.

In order for A to have a pivot in every row, it must have at least as many columns as rows: $m \le n$.

(1	0	*	0	*)
0	1	*	0	*
0/	0	0	1	*/

For instance, \mathbf{R}^2 is "too small" to map *onto* \mathbf{R}^3 .

Definition

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** (or **into**, or **injective**) if *different* vectors in \mathbb{R}^n map to different vectors in \mathbb{R}^m . In other words, each b in \mathbb{R}^m is the image of at most one x in \mathbb{R}^n : different inputs have different outputs. Note that not one-to-one means different vectors in \mathbb{R}^n have the same image.



Theorem

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- ► T is one-to-one
- T(x) = b has one or zero solutions for every b in \mathbf{R}^m
- Ax = b has a *unique solution or is inconsistent* for every b in \mathbf{R}^m
- Ax = 0 has a unique solution
- A has a pivot in every column.
- The columns of A are linearly independent

Question

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: A must have at least as many rows as columns $(n \le m)$ to have a pivot in every column.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$
 For instance, \mathbf{R}^4 is "too big" to map *into* \mathbf{R}^2

Extra: Linear Transformations are Matrix Transformations Recap

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T \colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = T\begin{pmatrix}x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}\end{pmatrix}$$
$$= T(xe_1 + ye_2 + ze_3)$$
$$= xT(e_1) + yT(e_2) + zT(e_3)$$
$$= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$= A\begin{pmatrix}x\\y\\z\end{pmatrix}.$$