Announcements

Wednesday, October 04

- Quiz **this** Friday covers sections 1.7,1.8 and 1.9.
- Quiz will have two questions

  Define $T(x) = Ax$ *with* $A = \ldots$. *Is the transformation... ? Provide...*

  Design a transformation $T : R^2 \to R^4$ *that satisfies...*

  Expectations:
- You need to know all new notation in those sections.
- And you need to understand how those concepts are related.
- Linear independence is also involved in those concepts.

Section 2.2

The Inverse of a Matrix

The Definition of Inverse

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is *written* A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Wild guess: $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

✔

Elementary Matrices

Definition

An **elementary matrix** is a matrix E that *differs* from I_n by one *row operation*.

There are **three kinds**, corresponding to the three elementary row operations:

scaling $(R_2 = 2R_2)$	row replacement $(R_2 = R_2 + 2R_1)$	swap $(R_1 \longleftrightarrow R_2)$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Important Fact: For any $n \times n$ matrix A, if E is the elementary matrix for a row operation, then *EA differs from A by the same row operation*.

Example:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

Inverse of Elementary Matrices

Elementary matrices are invertible. The inverse is the elementary matrix which un-does the row operation.

$R_2 = R_2 \times 2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$R_2 = R_2 \div 2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 = R_2 + 2R_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$R_2 = R_2 - 2R_1$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$R_1 \longleftrightarrow R_2$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Poll

Poll

Let E be the 3×3 matrix corresponding to *swapping rows* 1 and 3. Mark *both E and E^{-1}* from the list below

$$a) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad b) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$c) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad d) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution: Both E and E^{-1} are equal to $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Solving Linear Systems via Inverses

Theorem

If A is **invertible**, then for every b there is *unique solution* to $Ax = b$:

$$x = A^{-1}b.$$

Verify: Multiple by A on the left!

$$Ax = AA^{-1}b = I_n b = b$$

Example

Solve the system

could be any other vector

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x \phantom{{}+3y} + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Answer: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Computing A^{-1}

Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(\, A \mid I_n \,)$.
2. If the result has the form $(\, I_n \mid B \,)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

Computing A^{-1}

Example

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 \div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

So $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}$.

Check: $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✔

Why Does This Work?

First answer: We can think of the algorithm as *simultaneously solving* the equations

$$Ax_1 = e_1 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

- From theory: $x_i = A^{-1}Ax_i = A^{-1}e_i$. So *$x_i$ is the i-th column of A^{-1}*.
- *Row reduction*: the solution *x_i appears* in i-th column in the augmented part.

Second answer: Through *elementary matrices*, see extra material at the end.

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\boxed{\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.}$$

Fact

A is **invertible only when** $\det(A) \neq 0$, and

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We can **get the identity** only when $ad - bc \neq 0$. Verify:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Useful Facts

Suppose A, B and C are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Important: AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}\ \ B^{-1}A^{-1}$.

Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
Similarly, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I_n.$$

In general

The product of invertible matrices is invertible.
The *inverse is the product* of the inverses, in the *reverse order*.

Extra: Why Does The Inversion Algorithm Work?

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to I_n.

Why? Say the row operations taking A to I_n are the elementary matrices $E_1, E_2, \ldots, E_k$. So

pay attention to the order! $\longrightarrow$ $E_k E_{k-1} \cdots E_2 E_1 A = I_n$

$$\implies E_k E_{k-1} \cdots E_2 E_1 A A^{-1} = A^{-1}$$

$$\implies E_k E_{k-1} \cdots E_2 E_1 I_n = A^{-1}.$$

This is what we do when row reducing the augmented matrix:
Do same row operations to A (first line above) and to I_n (last line above).
Therefore, you'll end up with I_n and A^{-1}.

$$\left(A \mid I_n \right) \rightsquigarrow \left(I_n \mid A^{-1} \right)$$

Section 2.3

Characterization of Invertible Matrices

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that for all x in $\mathbf{R}^n$

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x.$$

In this case we say U is the **inverse** of T, and we **write** $U = T^{-1}$.

In other words, $T(U(x)) = x$, so *T "undoes" U*, and likewise U "undoes" T.

Fact

A transformation T is invertible if and only if *it is both one-to-one and onto*.

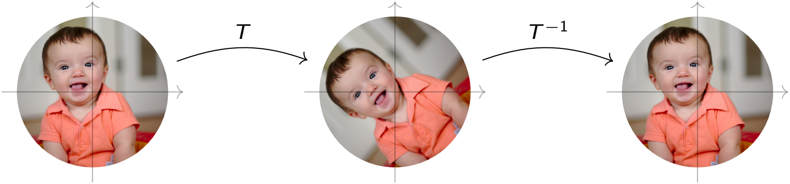
This means *for every y* in $\mathbf{R}^n$, *there is a unique x* in $\mathbf{R}^n$ such that $T(x) = y$.

Therefore we can define $T^{-1}(y) = x$.

Invertible Transformations

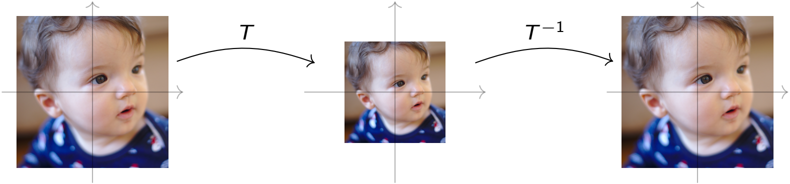
Examples

Let $T =$ **counterclockwise** rotation in the plane by 45°. What is T^{-1}?



T^{-1} is *clockwise* rotation by 45°.

Let $T =$ **shrinking** by a factor of $2/3$ in the plane. What is T^{-1}?



T^{-1} is *stretching* by $3/2$.

Invertible Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be **an invertible linear transformation** with matrix A.

Let B be the matrix for T^{-1}. We know $T \circ T^{-1}$ has matrix AB, so for all x,

$$ABx = T \circ T^{-1}(x) = x.$$

Hence $AB = I_n$, that is $B = A^{-1}$ (This is why we define matrix inverses).

Fact

If T *is an invertible* linear transformation with *matrix* A, then T^{-1} is an invertible linear transformation with **matrix** A^{-1}.

Non-invertibility: E.g. let $T =$ *projection* onto the x-axis. What is T^{-1}? *It is not invertible*: you can't undo it.

It's corresponding matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible!

Invertible transformations

Example 1

Let T = *shrinking by a factor of* $2/3$ in the plane. Its matrix is

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Then T^{-1} = *stretching by* $3/2$. Its matrix is

$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

Check:

$$AB = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

The matrix corresponding to $T \circ T^{-1}$ is AB, which satisfies $(AB)x = x$

Note: the matrix corresponding to $T^{-1} \circ T$ is BA, also satisfies $(BA)x = x$

Invertible transformations

Example 2

Let $T =$ counterclockwise *rotation* in the plane *by* 45°. Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $T^{-1} =$ counterclockwise *rotation by* -45°. Its matrix is

$$B = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Check:

$$AB = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \checkmark$$

The matrix corresponding to $T \circ T^{-1}$ is AB, which satisfies $(AB)x = x$ Note: the matrix corresponding to $T^{-1} \circ T$ is BA, also satisfies $(BA)x = x$

The Really Big Theorem for Square Matrices of Math 1553

The Invertible Matrix Theorem

Let A **be an** $n \times n$ **matrix**, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be defined by $T(x) = Ax$.
The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. T is one-to-one.
4. T is onto.
5. A has a left inverse (there exists B such that $BA = I_n$).
6. A has a right inverse (there exists B such that $AB = I_n$).
7. A^T is invertible.
8. A is row equivalent to I_n.
9. A has n pivots (one on each column and row).
10. The columns of A are linearly independent.
11. $Ax = 0$ has only the trivial solution.
12. The columns of A span $\mathbf{R}^n$.
13. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.

you really have to understand these

Approach to The Invertible Matrix Theorem

As with all Equivalence theorems:

- For **invertible matrices**: all statements of the Invertible Matrix Theorem are true.
- For **non-invertible matrices**: *all statements* of the Invertible Matrix Theorem *are false*.

Tackle the assertions!

You know enough at this point to be able to *reduce all* of the statements *to assertions about the pivots* of a square matrix.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and *try to figure out on your own* why they're all equivalent.