# Section 2.8

## Subspaces of $\mathbf{R}^n$

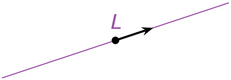
# Subspaces: Motivation and examples

### Example

The subset $\{0\}$:

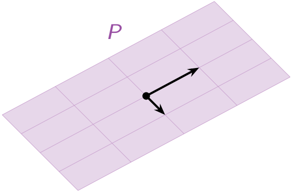
### Example

A line $L$ *through the origin*:




### Example

A plane $P$ *through the origin*:



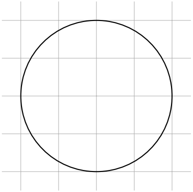

### Example

All of $\mathbf{R}^n$:

- The **span was our first example** of subspace: $Span\{v_1, \ldots, v_p\}$.
- But in general, subspaces are not defined by *'the generating vectors'*

# Subsets vs. Subspaces

A *subset* of $\mathbf{R}^n$ is *any collection* of vectors whatsoever.



Subset: *yes*
Subspace: *no*

A **subspace** is a special kind of subset, which satisfies *three defining properties:*

1. "not empty"
2. "closed under addition"
3. "closed under $\times$ scalars"

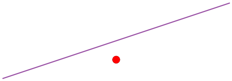
# Non-Examples

**Color code**

Purple: wanna-be 'subspaces'
Red vectors: **would have to be in** the subset too.

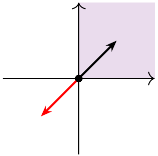
### Non-Example

Any set that *doesn't contain the origin*



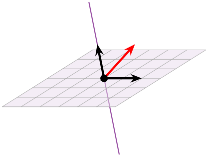
### Non-Example

The first quadrant in $\mathbf{R}^2$.



### Non-Example

A line union a plane in $\mathbf{R}^3$.



# The Definition of Subspace

### Definition

A **subspace** of $\mathbf{R}^n$ is a subset $V$ of $\mathbf{R}^n$ satisfying:

1. *The zero* vector *is in* $V$. "not empty"
2. If $u$ *and* $v$ are in $V$, *then* $u + v$ is also in $V$. "closed under addition"
3. If $u$ is in $V$ and $c$ is in $\mathbf{R}$, *then* $cu$ is in $V$. "closed under $\times$ scalars"

**Consequences** of definition:

- By (3), *if* $v$ is in $V$, then *so is the line through* $v$.
- By (2),(3), *if* $u, v$ are in $V$, then *so is* $xu + yv$, for all $x, y \in \mathbf{R}$.

A subspace $V$ *contains the span* of any set *of vectors in* $V$.

# Spans are Subspaces

### Theorem

Any $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is a subspace.

### Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_n\}$, we say that $V$ is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

!!!

Every span is a subspace but also every subspace is a span.

How would you *find the generating vectors*?

# Poll

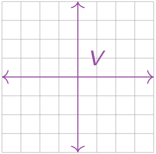
# Subspaces

## Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if $V$ is a subspace or not.

1. Does $V$ contain the zero vector?

3. Is $V$ closed under scalar multiplication?




2. Is $V$ closed under addition?

# Basis and dimension of a Subspace

Let $V$ be a subspace of $\mathbf{R}^n$ and $\{v_1, v_2, \ldots, v_m\}$ in $V$ linearly independent. So that every time you *'add one'* of these vectors, the *span gets bigger*.

What if $Span\{v_1, \ldots, v_m\} = V$?

Then any smaller set can't span $V$. If we *remove any vector*, the span gets *smaller*:

**Definition**

Let $V$ be a subspace of $\mathbf{R}^n$. A **basis** of $V$ is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $V$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is *linearly independent*.

The number of vectors in a basis is the **dimension** of $V$, and is written $\dim V$.

**Important**

A subspace has many *different bases*, but they all have the *same number* of vectors (see the exercises in §2.9).

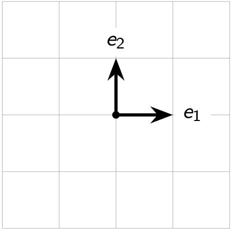
# Bases of $\mathbf{R}^2$

### Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.
2. They are linearly independent.

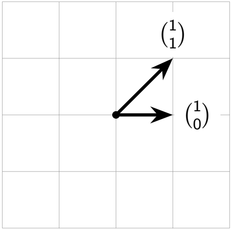



### Question

What is another basis for $\mathbf{R}^2$?

Any two *nonzero vectors* that are *not collinear*.
$\left\{\binom{1}{0}, \binom{1}{1}\right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



# Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: $I_n$ has a pivot in every row.
2. They are linearly independent: $I_n$ has a pivot in every column.

**In general:**

$\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if *A is invertible*.

## Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that *$\mathcal{B}$ is a basis* for $V$.

# Subspaces of a transformation

An $m \times n$ matrix $A$ naturally gives rise to *two* subspaces.

## Definition

The *column space* of $A$ is the subspace of $\mathbf{R}^m$ spanned by the columns of $A$. It is *written* $\operatorname{Col} A$.

The **null space** of $A$ is a subspace of $\mathbf{R}^n$ containing the set of all solutions of the homogeneous equation $Ax = 0$:

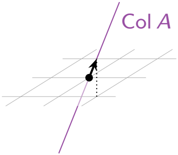
$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

# Column Space and Null Space

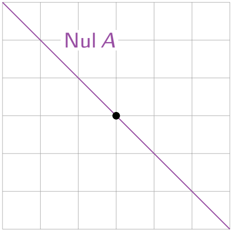
## Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the *column space*:




Let's compute the **null space**:



# Verify: The null space is a subspace and a span

Check that the null space is a subspace:

### Question

How to find vectors which span the null space? Answer: Parametric vector form!

We know that the *solution set to $Ax = 0$* has a parametric form that *looks like*

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, *say*, $x_3$ and $x_4$ are the *free variables*. So

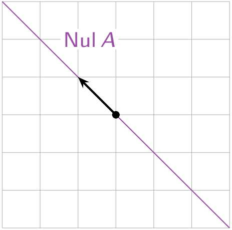
$$\text{Nul}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §1.5 (Solution Sets).

# Find Null Space as a Span

## Example

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.



## Basis for Nul $A$

**Fact**

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul $A$.

Example

# Basis for Col $A$

**Fact**

The *pivot columns* of $A$ always form a basis for Col $A$.

**Warning:** It is the pivot columns of the *original matrix* $A$, **not the row-reduced** form. (Row reduction changes the column space.)

Example

Why? End of §2.8, or ask in office hours.

**How do you check if a subset is a subspace?**

- Is it a span?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?

*Can it be written as*

- a span?
- the column space of a matrix?
- the null space of a matrix?
- a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

**If all else fails**:

- Can you *verify directly* that it satisfies the *three defining properties*?