Announcements

Monday, October 16

- **Midterm** this Friday October 20th:
  - covers all material (except Sections 2.4-2.7) **through Section 2.9**
  - *type of questions* from Sections 1.7 through 2.9
- Review Session this Wednesday. Which topics do you want to review?

Subspaces of a transformation

Recall: a **basis** of a subspace V is a set of vectors that

- *spans* V and
- is *linearly independent*.

Let A be an $m \times n$ matrix.

- The *column space* of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is *written* $\operatorname{Col} A$.
- The **null space** of A is a subspace of $\mathbf{R}^n$ containing the set of all solutions of the homogeneous equation $Ax = 0$:

$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

Fact

The *vectors in* the parametric vector form of the *general solution to* $Ax = 0$ always form a **basis for** Nul A.

Example

1. Every solution to $Ax = 0$ has this form. So the *vectors span* Nul A by construction.
2. Look at the *last two rows of the basis*. Can you see why *they are linearly independent*?

Basis for Col A

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: It is the pivot columns of the **original matrix** A, *not the row-reduced* form. (Row reduction changes the column space.)

Example

Why? End of §2.8, or ask in office hours.

Section 2.9

Dimension and Rank

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a *basis for V*.
- A **basis for the column space** of a matrix A is given by the *pivot columns*.
- A **basis for the null space** of A is given by the vectors attached to the *free variables* in the parametric vector form.

Definition

The **rank** of a matrix A, *written* $\operatorname{rank} A$, is the *dimension of the range* of $T(x) = Ax$ (dimension of $\operatorname{Col} A$).

Observe:

$$\begin{aligned} \operatorname{rank} A = \dim \operatorname{Col} A &= \text{the number of columns with pivots} \\ \dim \operatorname{Nul} A &= \text{the number of free variables} \\ &= \text{the number of columns without pivots.} \end{aligned}$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A

free variables

The Basis Theorem

Basis Theorem

Let V be a **subspace of dimension** m. Then:

- Any *m linearly independent* vectors in V form *a basis* for V.
- Any *m vectors that span* V form *a basis* for V.

Upshot

If you *already* know that $\dim V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then *check only one* of

1. $\mathcal{B}$ is linearly independent, or
2. $\mathcal{B}$ spans V

in order **for $\mathcal{B}$ to be a basis**.

Poll

Bases as Coordinate Systems

The unit coordinate vectors *$e_1, e_2, \ldots, e_n$ form a basis for $\mathbf{R}^n$*. Any vector is a *unique linear combination* of the e_i:

$$v = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2.$$

Note that the coordinates of v are exactly the coefficients of e_1, e_2, e_3.

Going backwards: for any basis $\mathcal{B}$, we *interpret the coefficients* of a linear combination as **coordinates with respect to $\mathcal{B}$**.

Definition

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace $V \subset \mathbf{R}^p$ (so $m \leq p$).

unique linear combination $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$.
The **$\mathcal{B}$-coordinate vector of x** is the vector

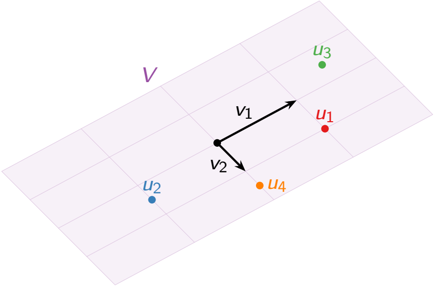
$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

These form *a basis $\mathcal{B}$* for the plane

$$V = \text{Span}\{v_1, v_2\} \text{in } \mathbf{R}^4.$$



Question: Estimate the *$\mathcal{B}$-coordinates* of these vectors:

Remark

Make sense of V as two-dim: Choose a basis $\mathcal{B}$ and use $\mathcal{B}$-coordinates.
Careful: The coordinates give *only the coefficients* of a linear combination *using such basis vectors*.

Bases as Coordinate Systems

Example 1

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathcal{B} = \{v_1, v_2\}, \quad V = \mathrm{Span}\{v_1, v_2\}.$

Verify that $\mathcal{B}$ is a basis:

Question: If $[x]_{\mathcal{B}} = \binom{5}{2}$, then what is x?

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

$$\text{Let } v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \mathrm{Span}\{v_1, v_2, v_3\}.$$

Question: Find a basis for V.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

Using the *Rank Theorem* and the *Basis Theorem*, we have new interpretations of the **meaning of invertibility**.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T: \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. *The following statements are equivalent.*

1. **A is invertible**.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$\boxed{[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.}$$

Finding the $\mathcal{B}$-coordinates for x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$

Question: What happens if you try to find the $\mathcal{B}$-coordinates of *x not in V*?

Extra: Why coefficients are unique

Lemma like a theorem, but less important

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ **is a basis** for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for **unique coefficients** $c_1, c_2, \ldots, c_m$.