## Announcements

Monday, October 16
$\triangleright$ Midterm this Friday October 20th:

- covers all material (except Sections 2.4-2.7) through Section 2.9
- type of questions from Sections 1.7 through 2.9
$\triangleright$ Review Session this Wednesday. Which topics do you want to review?


## Subspaces of a transformation

Recall: a basis of a subspace $V$ is a set of vectors that

- spans $V$ and
- is linearly independent.

Let $A$ be an $m \times n$ matrix.

- The column space of $A$ is the subspace of $\mathbf{R}^{m}$ spanned by the columns of $A$. It is written $\operatorname{Col} A$.
- The null space of $A$ is a subspace of $\mathbf{R}^{n}$ containing the set of all solutions of the homogeneous equation $A x=0$ :

$$
\operatorname{Nul} A=\left\{x \text { in } \mathbf{R}^{n} \mid A x=0\right\}
$$

## Basis for $\operatorname{Nul} A$

## Fact

The vectors in the parametric vector form of the general solution to $A x=0$ always form a basis for $\mathrm{Nul} A$.

Example

$$
A=\left(\begin{array}{rrrr}
1 & 2 & 0 & -1 \\
-2 & -3 & 4 & 5 \\
2 & 4 & 0 & -2
\end{array}\right) \xrightarrow{\text { rref }} \underset{\sim}{\text { r. }}\left(\begin{array}{rrrr}
1 & 0 & -8 & -7 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

$$
\underset{\substack{\text { parametric } \\
\text { vector } \\
\text { form } \\
\text { mann }}}{ } x=x_{3}\left(\begin{array}{c}
8 \\
-4 \\
1 \\
0
\end{array}\right)+x_{4}\left(\begin{array}{c}
7 \\
-3 \\
0 \\
1
\end{array}\right) \xrightarrow{\substack{\text { basis of } \\
\text { Nul } A \\
\text { mum }}}\left\{\left(\begin{array}{c}
8 \\
-4 \\
1 \\
0
\end{array}\right),\left(\begin{array}{c}
7 \\
-3 \\
0 \\
1
\end{array}\right)\right\}
$$

1. Every solution to $A x=0$ has this form.

So the vectors span $\operatorname{Nul} A$ by construction.
2. Look at the last two rows of the basis. Can you see why they are linearly independent?

## Basis for $\operatorname{Col} A$

## Fact

The pivot columns of $A$ always form a basis for $\operatorname{Col} A$.

Warning: It is the pivot columns of the original matrix $A$, not the row-reduced form. (Row reduction changes the column space.)

Example

So a basis for $\operatorname{Col} A$ is

$$
\left\{\left(\begin{array}{r}
1 \\
-2 \\
2
\end{array}\right),\left(\begin{array}{r}
2 \\
-3 \\
4
\end{array}\right)\right\}
$$

Why? End of $\S 2.8$, or ask in office hours.

## Section 2.9

Dimension and Rank

## The Rank Theorem

## Recall:

- The dimension of a subspace $V$ is the number of vectors in a basis for $V$.
- A basis for the column space of a matrix $A$ is given by the pivot columns.
- A basis for the null space of $A$ is given by the vectors attached to the free variables in the parametric vector form.

Definition
The rank of a matrix $A$, written rank $A$, is the dimension of the range of $T(x)=A x($ dimension of $\operatorname{Col} A)$.
Observe:

$$
\begin{aligned}
\operatorname{rank} A=\operatorname{dim} \operatorname{Col} A & =\text { the number of columns with pivots } \\
\operatorname{dim} \operatorname{Nul} A & =\text { the number of free variables } \\
& =\text { the number of columns without pivots. }
\end{aligned}
$$

## Rank Theorem

If $A$ is an $m \times n$ matrix, then
$\operatorname{rank} A+\operatorname{dim} \operatorname{Nul} A=n=$ the number of columns of $A$.

## The Rank Theorem

## Example

$$
A=(\underbrace{\left.\begin{array}{r}
1 \\
-2 \\
2
\end{array} \begin{array}{rrr}
2 \\
-3 & 0 & -1 \\
4 & 5 \\
0 & -2
\end{array}\right) \underset{\text { free variables }}{\text { muref }}\left(\begin{array}{ll}
1 & 0 \\
0 & 1 \\
0 & 0
\end{array} \begin{array}{r}
-8 \\
4 \\
0 \\
0
\end{array}\right)}_{\text {basis of } \operatorname{Col} A}
$$

A basis for $\operatorname{Col} A$ is

$$
\left\{\left(\begin{array}{r}
1 \\
-2 \\
2
\end{array}\right),\left(\begin{array}{r}
2 \\
-3 \\
4
\end{array}\right)\right\}
$$

so $\operatorname{rank} A=\operatorname{dim} \operatorname{Col} A=2$.
Since there are two free variables $x_{3}, x_{4}$, the parametric vector form for the solutions to $A x=0$ is

$$
x=x_{3}\left(\begin{array}{r}
8 \\
-4 \\
1 \\
0
\end{array}\right)+x_{4}\left(\begin{array}{r}
7 \\
-3 \\
0 \\
1
\end{array}\right) \underset{\underset{\text { minnumin }}{\text { basis }} \operatorname{Nul} A}{\text { bunn }}\left\{\left(\begin{array}{r}
8 \\
-4 \\
1 \\
0
\end{array}\right),\left(\begin{array}{r}
7 \\
-3 \\
0 \\
1
\end{array}\right)\right\} .
$$

Thus $\operatorname{dim} \operatorname{Nul} A=2$.
The Rank Theorem says $2+2=4$.

## The Basis Theorem

## Basis Theorem

Let $V$ be a subspace of dimension $m$. Then:

- Any $m$ linearly independent vectors in $V$ form a basis for $V$.
- Any $m$ vectors that span $V$ form a basis for $V$.

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Upshot
If you already know that dim}V=m\mathrm{ , and you have m
vectors }\mathcal{B}={\mp@subsup{v}{1}{},\mp@subsup{v}{2}{},\ldots,\mp@subsup{v}{m}{}}\mathrm{ in }V\mathrm{ , then check only one
of
    1. \mathcal{B is linearly independent, or}
    2. }\mathcal{B}\mathrm{ spans V
in order for \mathcal{B to be a basis.}
```


## Poll

## Poll

Let $A$ and $B$ be $3 \times 3$ matrices. Suppose that $\operatorname{rank}(A)=2$ and $\operatorname{rank}(B)=2$. Is it possible that $A B=0$ ?

Our information, by the rank theorem:
$\operatorname{rank}(A)=2$ and $\operatorname{dim} \operatorname{Nul} A=1$, also
$\operatorname{rank}(B)=\operatorname{dim} \operatorname{Col} B=2$ and $\operatorname{dim} \operatorname{Nul} B=1$.
If $A B=0$, then $A B x=0$ for every $x$ in $\mathbf{R}^{3}$.
This means $A(B x)=0$ for all $x \in \mathbf{R}^{3}$. Every vector $B x$ is in Nul $A$.
Then the range of $T(x)=B x$ (same as Col $B$ ) is contained in Nul $A$.
But a 1-dimensional space can't contain a 2-dimensional space.


## Bases as Coordinate Systems

The unit coordinate vectors $e_{1}, e_{2}, \ldots, e_{n}$ form a basis for $\mathbf{R}^{n}$. Any vector is a unique linear combination of the $e_{i}$ :

$$
v=\binom{3}{5}=3\binom{1}{0}+5\binom{0}{1}=3 e_{1}+5 e_{2} .
$$

Note that the coordinates of $v$ are exactly the coefficients of $e_{1}, e_{2}, e_{3}$.
Going backwards: for any basis $\mathcal{B}$, we interpret the coefficients of a linear combination as coordinates with respect to $\mathcal{B}$.
Definition
Let $\mathcal{B}=\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}$ be a basis of a subspace $V \subset \mathbf{R}^{p}$ (so $m \leq p$ ). The coordinates of $x$ with respect to $\mathcal{B}$ are the coefficients $c_{1}, c_{2}, \ldots, c_{m}$ of the unique linear combination $x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m}$.
The $\mathcal{B}$-coordinate vector of $x$ is the vector

$$
[x]_{\mathcal{B}}=\left(\begin{array}{c}
c_{1} \\
\vdots \\
c_{m}
\end{array}\right) \quad \text { in } \mathbf{R}^{m}
$$

## Bases as Coordinate Systems

Picture

Let

$$
v_{1}=\left(\begin{array}{r}
2 \\
-1 \\
0 \\
1
\end{array}\right) \quad v_{2}=\left(\begin{array}{r}
1 \\
0 \\
0 \\
-1
\end{array}\right)
$$

These form a basis $\mathcal{B}$ for the plane

$$
V=\operatorname{Span}\left\{v_{1}, v_{2}\right\} \text { in } \mathbf{R}^{4}
$$

Question: Estimate the $\mathcal{B}$-coordinates of these vectors:

$$
\left[u_{1}\right]_{\mathcal{B}}=\binom{1}{1} \quad\left[u_{2}\right]_{\mathcal{B}}=\binom{-1}{\frac{1}{2}} \quad\left[u_{3}\right]_{\mathcal{B}}=\binom{\frac{3}{2}}{-\frac{1}{2}} \quad\left[u_{4}\right]_{\mathcal{B}}=\binom{0}{\frac{3}{2}}
$$

Remark
Make sense of $V$ as two-dim: Choose a basis $\mathcal{B}$ and use $\mathcal{B}$-coordinates. Careful: The coordinates give only the coefficients of a linear combination using such basis vectors.

## Bases as Coordinate Systems

## Example 1

Let $v_{1}=\left(\begin{array}{l}1 \\ 0 \\ 1\end{array}\right), \quad v_{2}=\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right), \quad \mathcal{B}=\left\{v_{1}, v_{2}\right\}, \quad V=\operatorname{Span}\left\{v_{1}, v_{2}\right\}$.
Verify that $\mathcal{B}$ is a basis:
Span: by definition $V=\operatorname{Span}\left\{v_{1}, v_{2}\right\}$.
Linearly independent: because they are not multiples of each other.
Question: If $[x]_{\mathcal{B}}=\binom{5}{2}$, then what is $x$ ?

$$
[x]_{\mathcal{B}}=\binom{5}{2} \quad \text { means } \quad x=5 v_{1}+2 v_{2}=5\left(\begin{array}{l}
1 \\
0 \\
1
\end{array}\right)+2\left(\begin{array}{l}
1 \\
1 \\
1
\end{array}\right)=\left(\begin{array}{l}
7 \\
2 \\
7
\end{array}\right)
$$

Question: Find the $\mathcal{B}$-coordinates of $x=\left(\begin{array}{l}5 \\ 3 \\ 5\end{array}\right)$.
We have to solve the vector equation $x=c_{1} v_{1}+c_{2} v_{2}$ in the unknowns $c_{1}, c_{2}$.

$$
\left(\begin{array}{ll|l}
1 & 1 & 5 \\
0 & 1 & 3 \\
1 & 1 & 5
\end{array}\right) \text { ana }\left(\begin{array}{ll|l}
1 & 1 & 5 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{array}\right) \text { an } \rightarrow\left(\begin{array}{ll|l}
1 & 0 & 2 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{array}\right)
$$

So $c_{1}=2$ and $c_{2}=3$, so $x=2 v_{1}+3 v_{2}$ and $[x]_{\mathcal{B}}=\binom{2}{3}$.

## Bases as Coordinate Systems

## Example 2

Let $v_{1}=\left(\begin{array}{l}2 \\ 3 \\ 2\end{array}\right), v_{2}=\left(\begin{array}{c}-1 \\ 1 \\ 1\end{array}\right), v_{3}=\left(\begin{array}{l}2 \\ 8 \\ 6\end{array}\right), \quad V=\operatorname{Span}\left\{v_{1}, v_{2}, v_{3}\right\}$.
Question: Find a basis for $V$.
$V$ is the column span of the matrix

$$
A=\left(\begin{array}{rrr}
2 & -1 & 2 \\
3 & 1 & 8 \\
2 & 1 & 6
\end{array}\right) \underset{\sim}{\text { row reduce }}\left(\begin{array}{lll}
1 & 0 & 2 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{array}\right)
$$

A basis for the column span is formed by the pivot columns: $\mathcal{B}=\left\{v_{1}, v_{2}\right\}$.
Question: Find the $\mathcal{B}$-coordinates of $x=\left(\begin{array}{c}4 \\ 11 \\ 8\end{array}\right)$.
We have to solve $x=c_{1} v_{1}+c_{2} v_{2}$.

$$
\left(\begin{array}{rr|r}
2 & -1 & 4 \\
3 & 1 & 11 \\
2 & 1 & 8
\end{array}\right) \underset{\text { row reduce }}{\text { romm }}\left(\begin{array}{ll|l}
1 & 0 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{array}\right)
$$

So $x=3 v_{1}+2 v_{2}$ and $[x]_{\mathcal{B}}=\binom{3}{2}$.

## The Invertible Matrix Theorem

## Addenda

Using the Rank Theorem and the Basis Theorem, we have new interpretations of the meaning of invertibility.

The Invertible Matrix Theorem
Let $A$ be an $n \times n$ matrix, and let $T: \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ be the linear transformation $T(x)=A x$. The following statements are equivalent.

1. $A$ is invertible.
2. $T$ is invertible.
3. $A$ is row equivalent to $I_{n}$.
4. $A$ has $n$ pivots.
5. $A x=0$ has only the trivial solution.
6. The columns of $A$ are linearly independent.
7. $T$ is one-to-one.
8. $A x=b$ is consistent for all $b$ in $\mathbf{R}^{n}$.
9. The columns of $A$ span $\mathbf{R}^{n}$.
10. $T$ is onto.
11. $A$ has a left inverse (there exists $B$ such that $B A=I_{n}$ ).
12. $A$ has a right inverse (there exists $B$ such that $A B=I_{n}$ ).
13. $A^{T}$ is invertible.
14. The columns of $A$ form a basis for $\mathbf{R}^{n}$.
15. $\operatorname{Col} A=\mathbf{R}^{n}$.
16. $\operatorname{dim} \operatorname{Col} A=n$.
17. $\operatorname{rank} A=n$.
18. $\operatorname{Nul} A=\{0\}$.
19. $\operatorname{dim} \operatorname{Nul} A=0$.

## Bases as Coordinate Systems

If $\mathcal{B}=\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}$ is a basis for a subspace $V$ and $x$ is in $V$, then

$$
[x]_{\mathcal{B}}=\left(\begin{array}{c}
c_{1} \\
c_{2} \\
\vdots \\
c_{m}
\end{array}\right) \quad \text { means } \quad x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m}
$$

Finding the $\mathcal{B}$-coordinates for $x$ means solving the vector equation

$$
x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m}
$$

in the unknowns $c_{1}, c_{2}, \ldots, c_{m}$. This (usually) means row reducing the augmented matrix

$$
\left(\begin{array}{cccc|c}
\mid & \mid & & \mid & \mid \\
v_{1} & v_{2} & \cdots & v_{m} & x \\
\mid & \mid & & \mid & \mid
\end{array}\right)
$$

Question: What happens if you try to find the $\mathcal{B}$-coordinates of $\times$ not in $V$ ? You end up with an inconsistent system: $x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m}$ has no solution.

## Extra: Why coefficients are unique

## Lemma like a theorem, but less important

If $\mathcal{B}=\left\{v_{1}, v_{2}, \ldots, v_{m}\right\}$ is a basis for a subspace $V$, then any vector $x$ in $V$ can be written as a linear combination

$$
x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m}
$$

for unique coefficients $c_{1}, c_{2}, \ldots, c_{m}$.
Proof. We know $x$ is a linear combination of the $v_{i}$ (they span $V$ ). Suppose that we can write $x$ as a linear combination with different lists of coefficients:

$$
\begin{aligned}
& x=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{m} v_{m} \\
& x=c_{1}^{\prime} v_{1}+c_{2}^{\prime} v_{2}+\cdots+c_{m}^{\prime} v_{m}
\end{aligned}
$$

Subtracting:

$$
0=x-x=\left(c_{1}-c_{1}^{\prime}\right) v_{1}+\left(c_{2}-c_{2}^{\prime}\right) v_{2}+\cdots+\left(c_{m}-c_{m}^{\prime}\right) v_{m}
$$

Since $v_{1}, v_{2}, \ldots, v_{m}$ are linearly independent, they only have the trivial linear dependence relation. That means each $c_{i}-c_{i}^{\prime}=0$, or $c_{i}=c_{i}^{\prime}$.

