- Midterm this Friday October 20th:
  - covers all material (except Sections 2.4-2.7) through Section 2.9
  - type of questions from Sections 1.7 through 2.9

▷ Review Session this Wednesday. Which topics do you want to review?

Recall: a **basis** of a subspace V is a set of vectors that

- ▶ *spans* V and
- ► is linearly independent.

Let A be an  $m \times n$  matrix.

- ► The *column space* of A is the subspace of R<sup>m</sup> spanned by the columns of A. It is *written* Col A.
- ▶ The null space of A is a subspace of  $\mathbf{R}^n$  containing the set of all solutions of the homogeneous equation Ax = 0:

$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = \mathbf{0}\}.$$

### Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to Ax = 0 always form a **basis for** Nul A.

#### Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$parametric \\ vector \\ form \\ \text{vector} \\ form \\ \text{vector} \\ x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of}}_{\substack{\text{Nul } A \\ \text{vector}}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

- 1. Every solution to Ax = 0 has this form. So the *vectors span* Nul A by construction.
- 2. Look at the *last two rows of the basis*. Can you see why *they are linearly independent*?

### Basis for Col A

The *pivot columns* of *A* always form a basis for Col *A*.

Warning: It is the pivot columns of the original matrix *A*, not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -3 \\ 4 \\ 0 \\ -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis { pivot columns in rref

So a basis for Col A is

Fact

$$\left\{ \begin{pmatrix} 1\\ -2\\ 2 \end{pmatrix}, \begin{pmatrix} 2\\ -3\\ 4 \end{pmatrix} \right\}.$$

Why? End of §2.8, or ask in office hours.

## Section 2.9

Dimension and Rank

### The Rank Theorem

Recall:

- The dimension of a subspace V is the number of vectors in a *basis for* V.
- A basis for the column space of a matrix A is given by the *pivot columns*.
- ► A **basis for the null space** of *A* is given by the vectors attached to the *free variables* in the parametric vector form.

Definition

The **rank** of a matrix A, written rank A, is the dimension of the range of T(x) = Ax (dimension of Col A).

Observe:

rank  $A = \dim \text{Col } A = \text{the number of columns with pivots}$ dim Nul A = the number of free variables= the number of columns without pivots.

Rank Theorem If A is an  $m \times n$  matrix, then

rank  $A + \dim \operatorname{Nul} A = n =$ the number of columns of A.

# The Rank Theorem Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
  
basis of Col A free variables

A basis for Col A is

$$\left\{ \begin{pmatrix} 1\\-2\\2 \end{pmatrix}, \begin{pmatrix} 2\\-3\\4 \end{pmatrix} \right\},$$

so rank  $A = \dim \operatorname{Col} A = 2$ .

Since there are two free variables  $x_3$ ,  $x_4$ , the parametric vector form for the solutions to Ax = 0 is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus dim Nul A = 2.

The *Rank Theorem* says 2 + 2 = 4.

### The Basis Theorem

**Basis** Theorem

Let V be a subspace of dimension m. Then:

- ▶ Any *m* linearly independent vectors in *V* form *a* basis for *V*.
- Any *m* vectors that span V form a basis for V.

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Upshot

If you already know that dim V = m, and you have m vectors \mathcal{B} = \{v_1, v_2, \dots, v_m\} in V, then check only one of

1. \mathcal{B} is linearly independent, or

2. \mathcal{B} spans V

in order for \mathcal{B} to be a basis.
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Poll Let A and B be  $3 \times 3$  matrices. Suppose that rank(A) = 2 and rank(B) = 2. Is it **possible that** AB = 0?

Our information, by the rank theorem: rank(A) = 2 and dim Nul A = 1, also rank(B) =dim Col B = 2 and dim Nul B = 1.

If AB = 0, then ABx = 0 for every x in  $\mathbb{R}^3$ .

This means A(Bx) = 0 for all  $x \in \mathbf{R}^3$ . Every vector Bx is in Nul A.

Then the range of T(x) = Bx (same as Col *B*) is contained in Nul *A*. But a 1-dimensional space can't contain a 2-dimensional space.



### Bases as Coordinate Systems

The unit coordinate vectors  $e_1, e_2, \ldots, e_n$  form a basis for  $\mathbb{R}^n$ . Any vector is a *unique linear combination* of the  $e_i$ :

$$v = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2.$$

Note that the coordinates of v are exactly the coefficients of  $e_1, e_2, e_3$ .

Going backwards: for any basis  $\mathcal{B}$ , we *interpret the coefficients* of a linear combination as **coordinates with respect to**  $\mathcal{B}$ .

#### Definition

Let  $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$  be a basis of a subspace  $V \subset \mathbb{R}^p$  (so  $m \le p$ ). The coordinates of x with respect to  $\mathcal{B}$  are the coefficients  $c_1, c_2, \dots, c_m$  of the unique linear combination  $x = c_1v_1 + c_2v_2 + \dots + c_mv_m$ .

The  $\mathcal{B}$ -coordinate vector of x is the vector

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

### Bases as Coordinate Systems

Picture

Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$
  $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ 

These form a basis  $\mathcal{B}$  for the plane

$$V = \operatorname{Span}\{v_1, v_2\} \text{in } \mathbf{R}^4.$$



Question: Estimate the *B*-coordinates of these vectors:

$$[\mathbf{u}_1]_{\mathcal{B}} = \begin{pmatrix} 1\\1 \end{pmatrix} \qquad [\mathbf{u}_2]_{\mathcal{B}} = \begin{pmatrix} -1\\\frac{1}{2} \end{pmatrix} \qquad [\mathbf{u}_3]_{\mathcal{B}} = \begin{pmatrix} \frac{3}{2}\\-\frac{1}{2} \end{pmatrix} \qquad [\mathbf{u}_4]_{\mathcal{B}} = \begin{pmatrix} 0\\\frac{3}{2} \end{pmatrix}$$

### Remark

Make sense of V as two-dim: Choose a basis  $\mathcal{B}$  and use  $\mathcal{B}$ -coordinates. Careful: The coordinates give *only the coefficients* of a linear combination *using such basis vectors*.

### Bases as Coordinate Systems Example 1

Let 
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
,  $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ,  $\mathcal{B} = \{v_1, v_2\}$ ,  $V = \mathsf{Span}\{v_1, v_2\}$ .

Verify that  $\mathcal{B}$  is a basis: *Span*: by definition  $V = \text{Span}\{v_1, v_2\}$ . *Linearly independent*: because they are not multiples of each other.

Question: If  $[x]_{\mathcal{B}} = {5 \choose 2}$ , then what is x?

$$[x]_{\mathcal{B}} = \begin{pmatrix} 5\\2 \end{pmatrix} \quad \text{means} \quad x = 5v_1 + 2v_2 = 5\begin{pmatrix} 1\\0\\1 \end{pmatrix} + 2\begin{pmatrix} 1\\1\\1 \end{pmatrix} = \begin{pmatrix} 7\\2\\7 \end{pmatrix}.$$
  
Question: Find the  $\mathcal{B}$ -coordinates of  $x = \begin{pmatrix} 5\\3\\5 \end{pmatrix}$ .

. .

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We have to solve the vector equation  $x = c_1v_1 + c_2v_2$  in the unknowns  $c_1, c_2$ .

$$\begin{pmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 3 \\ 1 & 1 & | & 5 \end{pmatrix} \xrightarrow{} \begin{pmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{} \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{pmatrix}$$
  
So  $c_1 = 2$  and  $c_2 = 3$ , so  $x = 2v_1 + 3v_2$  and  $[x]_{\mathcal{B}} = \binom{2}{3}$ .

### Bases as Coordinate Systems Example 2

Let 
$$v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$
,  $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ,  $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$ ,  $V = \text{Span}\{v_1, v_2, v_3\}$ .

Question: Find a basis for V. V is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns:  $\mathcal{B} = \{v_1, v_2\}$ .

Question: Find the 
$$\mathcal{B}$$
-coordinates of  $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$ .

We have to solve  $x = c_1 v_1 + c_2 v_2$ .

$$\begin{pmatrix} 2 & -1 & | & 4 \\ 3 & 1 & | & 11 \\ 2 & 1 & | & 8 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix}$$

So  $x = 3v_1 + 2v_2$  and  $[x]_{\mathcal{B}} = \binom{3}{2}$ .

### The Invertible Matrix Theorem

Addenda

Using the *Rank Theorem* and the *Basis Theorem*, we have new interpretations of the **meaning of invertibility**.

### The Invertible Matrix Theorem

Let A be an  $n \times n$  matrix, and let  $T : \mathbf{R}^n \to \mathbf{R}^n$  be the linear transformation T(x) = Ax. The following statements are equivalent.

### 1. A is invertible.

- 2. T is invertible.
- 3. A is row equivalent to  $I_n$ .
- 4. A has n pivots.
- 5. Ax = 0 has only the trivial solution.
- 6. The columns of A are linearly independent.
- 7. T is one-to-one.

- 8. Ax = b is consistent for all b in  $\mathbb{R}^n$ .
- 9. The columns of A span  $\mathbb{R}^n$ .
- 10. T is onto.
- 11. A has a left inverse (there exists B such that  $BA = I_n$ ).
- 12. A has a right inverse (there exists B such that  $AB = I_n$ ).
- 13.  $A^T$  is invertible.
- 14. The columns of A form a basis for  $\mathbf{R}^n$ .

**15**. Col  $A = \mathbf{R}^{n}$ .

- 16. dim Col A = n.
- 17. rank A = n.
- 18. Nul  $A = \{0\}$ .
- **19**. dim Nul A = 0.

### Bases as Coordinate Systems

#### Summary

If  $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$  is a basis for a subspace V and x is in V, then  $[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m.$ Finding the  $\mathcal{B}$ -coordinates for x means solving the vector equation  $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ in the unknowns  $c_1, c_2, \ldots, c_m$ . This (usually) means row reducing the augmented matrix  $\left(\begin{array}{cccccccc} | & | & | & | & | \\ v_1 & v_2 & \cdots & v_m & | \\ | & | & | & | & | \end{array}\right).$ 

Question: What happens if you try to find the  $\mathcal{B}$ -coordinates of x not in V? You end up with an *inconsistent system*:  $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$  has no solution. Lemma like a theorem, but less important If  $\mathcal{B} = \{v_1, v_2, \dots, v_m\}$  is a basis for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for unique coefficients  $c_1, c_2, \ldots, c_m$ .

**Proof.** We know x is a linear combination of the  $v_i$  (they span V). Suppose that we can write x as a linear combination with different lists of coefficients:

$$x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$$
  
$$x = c'_1 v_1 + c'_2 v_2 + \dots + c'_m v_m$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \dots + (c_m - c_m')v_m$$

Since  $v_1, v_2, \ldots, v_m$  are linearly independent, they only have the trivial linear dependence relation. That means each  $c_i - c'_i = 0$ , or  $c_i = c'_i$ .