Announcements

Monday, October 23

- Webwork due next week,
- No quiz this week.

Chapter 3

Determinants

Section 3.1

Introduction to Determinants

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is **determinants**.

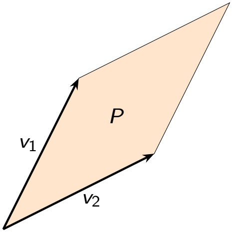
This is a completely *magical function* that takes a square matrix and gives you a number.
It is a very complicated functionthe formula for the determinant of a 10×10 matrix has $3,628,800$ summands! so we need efficient ways to compute it.

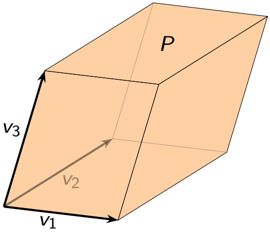
Today is mostly about the *computation* of determinants; in the next lecture we will focus on the theory.

The Idea of Determinants

Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Observation: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

The **determinant** of A will be a number $\det(A)$ whose absolute value is the *volume of P*.

Determinants of 2×2 Matrices

Revisited

We already have a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

What does this have to do with volumes?

The area of the parallelogram is always $|ad - bc|$.

Note: this shows $\det(A) \neq 0 \iff A$ is invertible in this case.

Determinants of 3×3 Matrices

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

How to remember this?

Draw a bigger matrix, repeating the first two columns to the right:

$$+\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

What does this have to do with *volumes*? *Next time.*

A Formula for the Determinant

When $n \geq 4$, the determinant is **not that simple to describe**. The formula is recursive:

We need some notation. Let A be an $n \times n$ matrix.

$$\begin{aligned} A_{ij} &= ij\text{th } \textbf{minor} \text{ of } A \\ &= (n-1)\times(n-1) \text{ matrix you get by } \textit{deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$C_{ij} = ij\text{th } \textbf{cofactor} \text{ of } A = (-1)^{i+j} \det A_{ij}$$

The signs of the cofactors follow a *checkerboard pattern*:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

Definition

The **determinant** of an $n \times n$ matrix A is

$$\det(A) = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

This formula is called **cofactor expansion** *along the first row*.

A Formula for the Determinant

1×1 Matrices

This is the beginning of the recursion.

$$\det(\, a_{11} \,) = a_{11}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The minors are:

$$A_{11} = \qquad\qquad A_{12} =$$

$$A_{21} = \qquad\qquad A_{22} =$$

The cofactors are

$$C_{11} = \qquad\qquad C_{12} =$$
$$C_{21} = \qquad\qquad C_{22} =$$

The determinant is

$$\det A = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

A Formula for the Determinant

3×3 Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The top row minors and cofactors are:

$A_{11} =$ $C_{11} =$

$A_{12} =$ $C_{12} =$

$A_{13} =$ $C_{13} =$

A Formula for the Determinant

Example

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} =$$

Cofactor expasion: Specify point of reference...

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

is called cofactor expansion *along the first row.* Actually, you can expand cofactors along any row or column you like!

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } j$$

Good trick: Use cofactor expansion along a row or a column *with a lot of zeros*.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A =$$

Poll

Trick: Expand along the first row

This works for any matrix that is *upper-triangular*
(all entries below the main diagonal are zero).

Theorem

The determinant of an upper-triangular matrix is the product of the diagonal entries:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

The same is true for lower-triangular matrices. (Repeatedly expand along the first row.)

Extra: A Formula for the Inverse

from §3.3

For 2×2 matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}.$$

Theorem

This last formula works for any $n \times n$ invertible matrix A:

(3, 1) entry

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det A} \left(C_{ij} \right)^T$$

Note that the cofactors are "transposed": the (i, j) entry of the matrix is C_{ji}.

The proof uses Cramer's rule. See Dan Margalit's notes on the website for a nice explanation.

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The minors are:

The cofactors are (don't forget to multiply by $(-1)^{i+j}$):

The determinant is (expanding along the first row):

$$\det A =$$

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The inverse is

$$A^{-1} =$$

Check: