Section 3.2

Properties of Determinants

Plan for Today

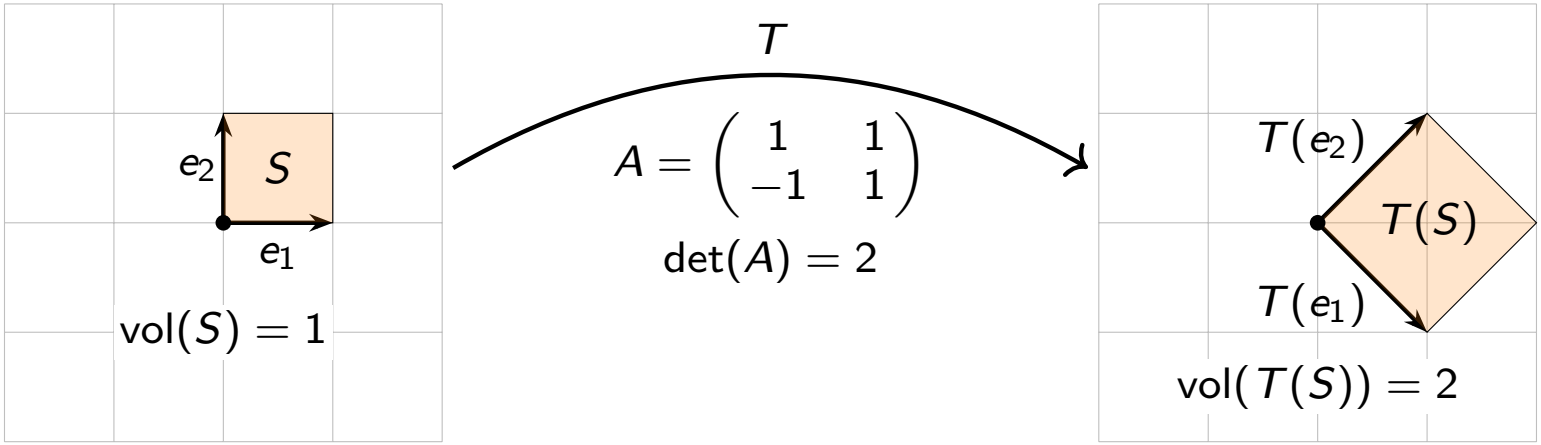
Last time, we gave a recursive formula for determinants in terms of cofactor expansions.

Plan for today:

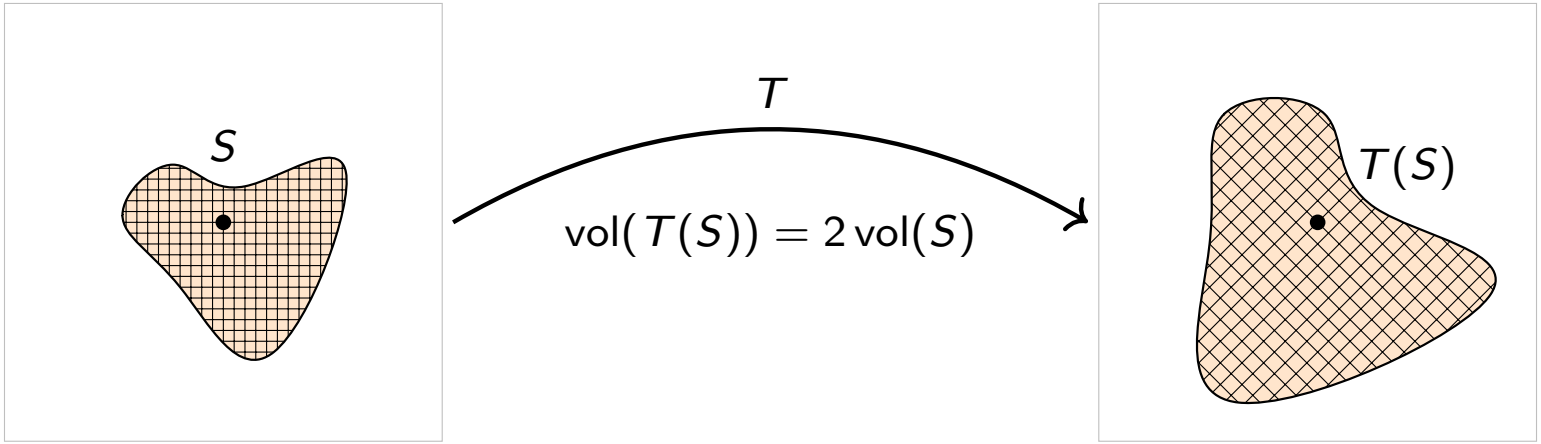
- An **abstract definition** of the determinant in terms of its properties.
- Computing determinants **using row operations**.
- Determinants and *products*: $\det(AB) = \det(A)\det(B)$,
- interpretation as volume,
- and linear transformations.

Linear Transformations and volumen

If S is the *unit cube*, then $T(S)$ is the parallelepiped formed by the columns of A. The **volumen changes** according to $\det(A)$.



For curvy regions: break S up into *tiny cubes*; each one is scaled by $|\det(A)|$. Then use *calculus* to reduce to the previous situation!



The Determinant is a Function

We can think of the *determinant as a function* of the entries of a matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{aligned} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{aligned}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms.

When mathematicians encounter a function whose *formula is too difficult* to write down, we try to **characterize it in terms of its properties**.

P characterizes object X

Not only does object X have property P,
but X **is the only one** thing that has property P.

Other example:

- e^x is unique function that has $f'(x) = f(x)$ and $f(0) = 1$.

Defining the Determinant in Terms of its Properties

Definition

The **determinant** is a function

$$\det\colon \{\text{square matrices}\} \longrightarrow \mathbf{R}$$

with the following **defining properties**:

1. $\det(I_n) = 1$
2. If we do a *row replacement* on a matrix, the determinant does not change.
3. If we *swap* two rows of a matrix, the determinant scales by -1.
4. If we *scale a row* of a matrix by k, the determinant scales by k.

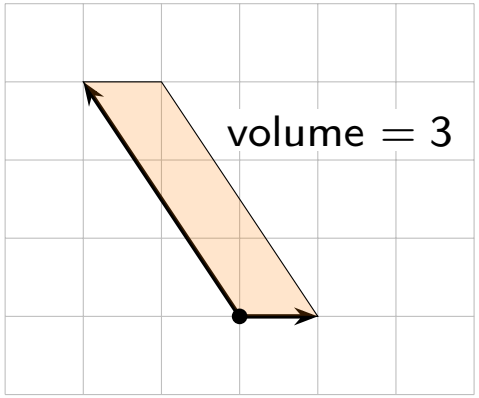
Why would we think of these properties? This is how volumes work!

1. The volume of the *unit cube* is 1.
2. Volumes don't change under *a shear*.
3. Volume of a *mirror image* is negative of the volume?
4. If you *scale one coordinate* by k, the volume is multiplied by k.

Properties of the Determinant

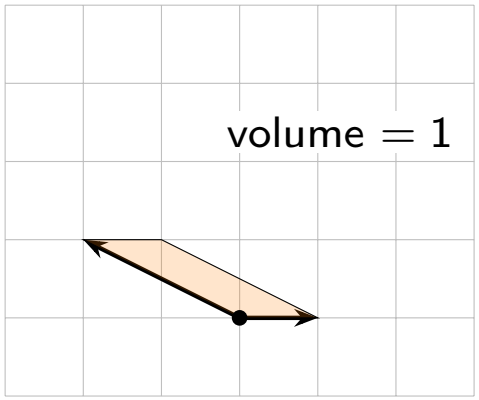
2×2 matrix

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$



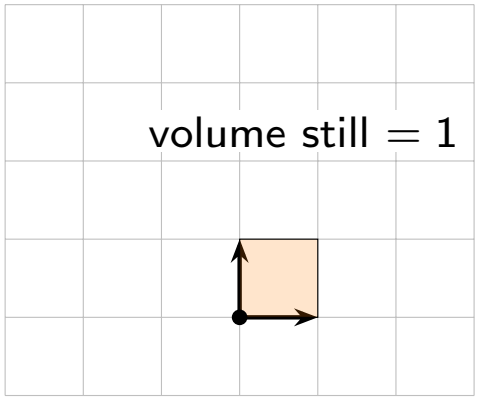
Scale: $R_2 = \frac{1}{3} R_2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



Row replacement: $R_1 = R_1 + 2R_2$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$



Determinant for Elementary matrices

It is easy to calulate the determinant of an elementary matrix:

$$\det\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\det\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

Poll

Computing the Determinant by Row Reduction

Example first

We can use the properties of the determinant and row reduction to compute the determinant of any matrix!

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} =$$

Computing the Determinant by Row Reduction

We can use the properties of the determinant and row reduction to compute the determinant of any matrix!

Recall: The determinant of a *triangular matrix* is the product of the diagonal entries.

Saving some work We can stop row reducing when we get to row echelon form.

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} = \cdots = -\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix} = 9.$$

Row reduction

This is almost always the easiest way to compute the determinant of a large, complicated matrix, either by hand or by computer.

Cofactor expansion is $O(n!) \sim O(n^n\sqrt{n})$, row reduction is $O(n^3)$.

Magical Properties of the Determinant

you really have to know these

1. $\det\colon \{\textbf{square matrices}\} \to \mathbf{R}$ is *the only function* satisfying the defining properties (1)–(4).
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A without row scaling, then
$$\det(A) = (-1)^{\#\text{swaps}}\big(\text{product of diagonal entries in REF}\big).$$
4. The determinant can be computed using any cofactor expansion.
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)
9. The determinant is multi-linear (we'll talk about this next).

Multi-Linearity of the Determinant

Think of det as a function of the *columns* of an $n \times n$ matrix:

$$\det\colon \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{n \text{ times}} \longrightarrow \mathbf{R}$$

$$\det(v_1, v_2, \ldots, v_n) = \det \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Multi-linear: For *any i* and any vectors $v_1, v_2, \ldots, v_n$ and *v_i' and any scalar c*,

$$\det(v_1, \ldots, v_i + v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$$
$$\det(v_1, \ldots, cv_i, \ldots, v_n) = c \det(v_1, \ldots, v_i, \ldots, v_n).$$

- We already knew: Scaling *one column* by c scales det by c.
- *This only works one column at a time.*
- *Same properties* hold if we replace column *by row*.

Extra: Mathematical intricacies

The characterization of the determinant function in terms of its properties is very useful. It will *give us a fast way to compute* determinants and prove the other properties.

The **disadvantage** of defining a function by its properties *before having a formula*:

- how do you know such a *function exists*?
- *is there only one* function satisfying those properties?

Extra: Intricacy applied

Why is Property 5 true? In Lay, there's a proof using elementary matrices. Here's another one.