Section 5.2

The Characteristic Equation

The Characteristic Polynomial

Last section we learn that for a square matrix A:

$$\begin{aligned}\lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ has a nontrivial solution} \\ &\iff (A - \lambda I)x = 0 \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is not invertible} \\ &\iff \det(A - \lambda I) = 0.\end{aligned}$$

Compute Eigenvalues

The *eigenvalues* of A are **the roots** of $\det(A - \lambda I)$

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Definition

The **trace** of a square matrix A is $\mathrm{Tr}(A) =$ sum of the diagonal entries of A.

What do you notice about: the characteristic polynomial of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$?

Shortcut

The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

Question: What are the eigenvalues of the *rabbit population matrix*

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Algebraic Multiplicity

Definition

The **algebraic multiplicity** of an eigenvalue λ is its *multiplicity as a root* of the characteristic polynomial.

Example

In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$. The algebraic multiplicities are

$$\lambda = \begin{cases} 2 & \text{multiplicity 1} \\ , -1 & \text{multiplicity 2} \end{cases}$$

Example

In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 + 2\sqrt{2}))(\lambda - (3 - 2\sqrt{2}))$. The algebraic multiplicities are $\lambda = \begin{cases} 3 + 2\sqrt{2} & \text{alg. multiplicity 1} \\ , 3 - 2\sqrt{2} & \text{alg. multiplicity 1} \end{cases}$

Multiplicities

Theorem

If A is an $n \times n$ matrix, the **characteristic polynomial**

$$f(\lambda) = \det(A - \lambda I)$$

is a polynomial of degree n, and its roots are the eigenvalues of A:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

Complex numbers

If you **count the eigenvalues** of A, with their algebraic multiplicities, depending on *whether you allow complex eigenvalues*, you will get :

- **Do allow** complex numbers: *Always n*.
- Only real numbers: Always at most n, but *sometimes less*.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix C such that

$$A = CBC^{-1}.$$

The intuition

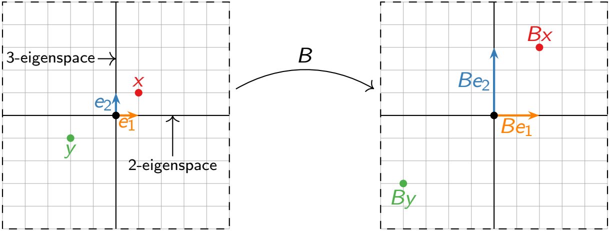
C keeps record of a basis $\mathcal{C} = \{v_1, \ldots, v_n\}$ of $\mathbf{R}^n$.

B transforms the $\mathcal{C}$-coordinates of x: $B[x]_{\mathcal{C}} = [Ax]_{\mathcal{C}}$
in *the same way that* A transforms the standard coordinates of x

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \implies \quad A = CBC^{-1}.$$

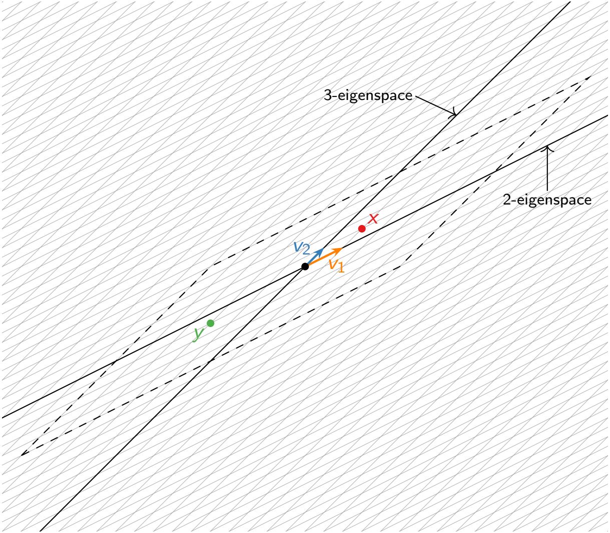
What does B do geometrically? **Scaling:** x-direction by 2 and y-direction by 3.

B acting on the usual coordinates



Now A will do to the standard coordinates what B does to the $\mathcal{C}$-coordinates, where $\mathcal{C} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

From $\mathcal{C}$-coordinates to standard coordinates



$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Bigg\} \text{ vectors in } \mathcal{C}$$

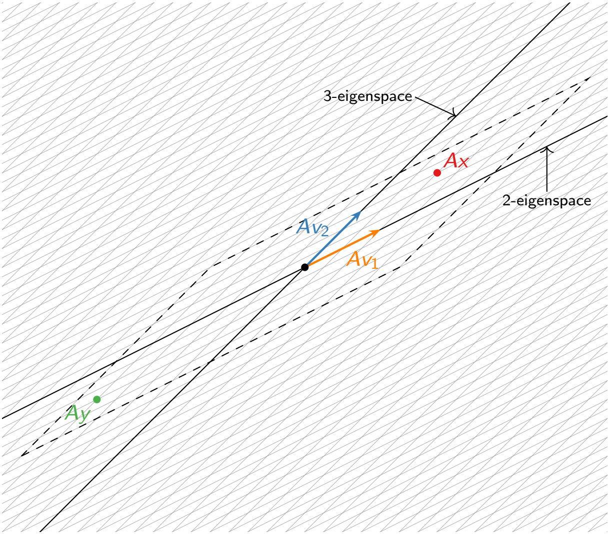
$$[x]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x =$$

$$[y]_{\mathcal{C}} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$y =$$

A does to the usual coordinates what B does to the $\mathcal{C}$-coordinates



$Av_1 =$

$Av_2 =$

$B[x]_{\mathcal{C}} = \qquad = [Ax]_{\mathcal{C}}$

$Ax =$

$B[y]_{\mathcal{C}} = \qquad = [Ay]_{\mathcal{C}}$

$Ay =$

Check:

Similar Matrices Have the Same Characteristic Polynomial

Fact

If A and B are **similar**, then they have the *same characteristic polynomial*.

Consequence: similar matrices have the *same eigenvalues*! Though different eigenvectors in general.

Why? Suppose $A = CBC^{-1}$. We can show that $\det(A - \lambda I) = \det(B - \lambda I)$.

Similarity Caveats

Warning

1. Matrices with the *same eigenvalues* need not be similar.
   For instance,
   $$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$
   both only have the eigenvalue 2, but they are *not similar*.
2. Similarity is *lost in row equivalence*.
   For instance,
   $$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
   are row equivalent, but they have *different eigenvalues*.

Once more: The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. *The following statements are equivalent.*

1. A is **invertible**.

2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

19. The determinant of A is *not equal to zero*.
20. The number 0 is *not an eigenvalue* of A.