

Announcements

Monday, November 06

- ▶ This week's quiz: covers *Sections 5.1 and 5.2*
- ▶ **Midterm 3**, on November 17th (next Friday)
 - ▶ Exam covers: Sections 3.1,3.2,5.1,5.2,5.3 and 5.5

Section 5.3

Diagonalization

Motivation: Difference equations

Now do multiply matrices

Many real-word (linear algebra problems):

- ▶ Start with a *given situation* (v_0) and
- ▶ want to know **what happens after some time** (iterate a transformation):

$$v_n = Av_{n-1} = \dots = A^n v_0.$$

- ▶ Ultimate question: *what happens in the long run* (find v_n as $n \rightarrow \infty$)

Old Example

Recall our example about *rabbit populations*:
using eigenvectors was **easier than matrix
multiplications**, but ...

- ▶ Taking *powers of diagonal* matrices is easy!
- ▶ Working with *diagonalizable matrices* is also easy.

Powers of Diagonal Matrices

If D is diagonal

Then D^n is also diagonal, the diagonal entries of D^n are the *n th powers of the diagonal* entries of D

Powers of Matrices that are Similar to Diagonal Ones

When is A is not diagonal?

Example

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Compute A^n . Using that

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

From the first expression:

$$A^2 =$$

$$A^3 =$$

$$\vdots$$

$$A^n =$$

Plug in P and D :

$$A^n =$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = PDP^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

$$A^k = PD^k P^{-1} = P \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} P^{-1}.$$

So diagonalizable matrices are *easy to raise to any power*.

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is **diagonalizable** if and only if A has n *linearly independent eigenvectors*.

In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where v_1, v_2, \dots, v_n are linearly independent **eigenvectors**, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the *corresponding eigenvalues* (in the same order).

Important

- ▶ If A has n *distinct eigenvalues* then A is **diagonalizable**.
- ▶ If A is diagonalizable matrix it *need not have n distinct eigenvalues* though.

Diagonalization

Example

Problem: Diagonalize $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

Diagonalization

Example 2

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Diagonalization

Example 2, continued

In this case: there are 3 linearly independent eigenvectors and only 2 distinct eigenvalues.

Diagonalization

Procedure

How to **diagonalize a matrix** A :

1. *Find the eigenvalues* of A using the characteristic polynomial.
2. *Compute a basis* \mathcal{B}_λ for each λ -eigenspace of A .
3. If there are **fewer than n total vectors** in the union of all of the eigenspace bases \mathcal{B}_λ , then the matrix is **not diagonalizable**.
4. *Otherwise*, the n vectors v_1, v_2, \dots, v_n in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

$$P = \left(\begin{array}{c|c|c|c} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{array} \right) \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue for v_i .

Diagonalization

A non-diagonalizable matrix

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is **not diagonalizable**.

Conclusion:

- ▶ All eigenvectors of A are multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- ▶ So A has only one linearly independent eigenvector
- ▶ If A was diagonalizable, there would be *two linearly independent eigenvectors*!

Non-Distinct Eigenvalues

Definition

Let λ be an eigenvalue of a square matrix A . The **geometric multiplicity** of λ is the *dimension of the λ -eigenspace*.

Theorem

Let λ be an eigenvalue of a square matrix A . Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

- ▶ **Note:** If λ is an eigenvalue, then the λ -eigenspace has dimension at least 1.
- ▶ ...but it might be smaller than what the characteristic polynomial suggests. The intuition/visualisation is *beyond the scope of this course*.

Non-Distinct Eigenvalues

(Good) examples

From *previous exercises* we know:

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The matrix $B = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = (1 - \lambda)(4 - \lambda) + 2 = (\lambda - 2)(\lambda - 3).$$

Matrix A	Geom. M.	Alg. M.
$\lambda = 1$	2	2
$\lambda = 2$	1	1

Matrix B	Geom. M.	Alg. M.
$\lambda = 2$	1	1
$\lambda = 3$	1	1

Thus, *both matrices are diagonalizable*.

Non-Distinct Eigenvalues

(Bad) example

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

We showed before that the *1-eigenspace has dimension 1* and A was **not diagonalizable**.

Eigenvalue	Geometric	Algebraic
$\lambda = 1$	1	2

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is **diagonalizable**.
2. The *sum of the geometric multiplicities* of the eigenvalues of A equals n .
3. The sum of all algebraic multiplicities is n . And for each eigenvalue, the *geometric and algebraic* multiplicity are equal.

Applications to Difference Equations

Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Start with a vector v_0 , and let $v_1 = Dv_0$, $v_2 = Dv_1, \dots, v_n = D^n v_0$.

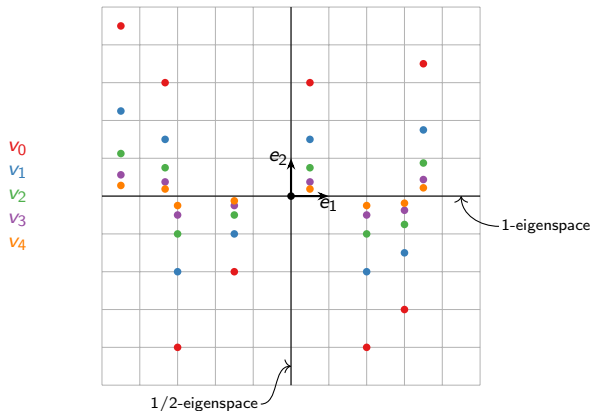
Question: What happens to the v_i 's for different starting vectors v_0 ?

- ▶ the x -coordinate equals the initial coordinate,
- ▶ the y -coordinate gets halved every time.

Applications to Difference Equations

Picture

$$D \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2 \end{pmatrix}$$



So all vectors get *"collapsed into the x-axis"*, which is the 1-eigenspace.

Applications to Difference Equations

More complicated example

$$\text{Let } A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

Start with a vector v_0 , and let $v_1 = Av_0$, $v_2 = Av_1, \dots$, $v_n = A^n v_0$.

Question: What happens to the v_i 's for different starting vectors v_0 ?

Matrix Powers: This is a diagonalization question. Bottom line: $A = PDP^{-1}$ for

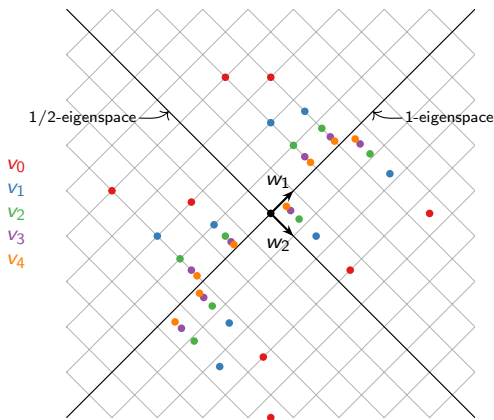
$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Hence $v_n = PD^n P^{-1} v_0$.

Applications to Difference Equations

Picture of the more complicated example

$A^n = PD^nP^{-1}$ acts on the usual coordinates of v_0 in the same way that D^n acts on the \mathcal{B} -coordinates, where $\mathcal{B} = \{w_1, w_2\}$.



So all vectors get "collapsed into the 1-eigenspace".

Extra: Proof Diagonalization Theorem

Why is the Diagonalization Theorem true?