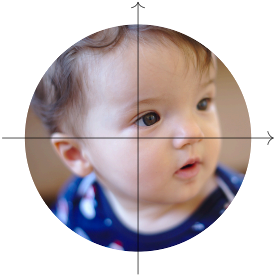
Section 5.5

Complex Eigenvalues

Motivation: Describe rotations

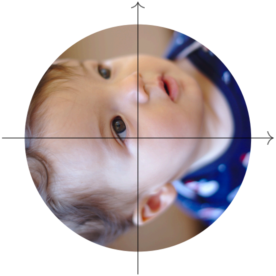
Among transformations, *rotations are very simple* to describe geometrically.

Where are the eigenvectors?



$$\xrightarrow{A}$$

no nonzero vector x
is collinear with Ax



The corresponding matrix has no real eigenvalues.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad f(\lambda) = \lambda^2 + 1.$$

Complex Numbers

Definition

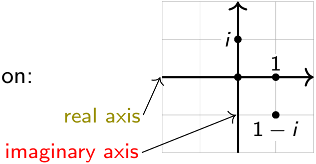
The **number** i is defined such that $i^2 = -1$.

Now we have to allow all possible combinations $a + b\,i$

Definition

A **complex number** is a number of the form $a + bi$ *for* a, b *in* $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

A picture of $\mathbf{C}$ uses a plane representation:



- $\mathbf{C}$ expands $\mathbf{R}$: real numbers are written $a + 0i$.
- Vector representation of $a + bi$: 2-coordinates $\binom{a}{b}$.
- What does it mean that $\binom{1}{0}$ and $\binom{0}{1}$ are linearly independent?

Operations on Complex Numbers (I)

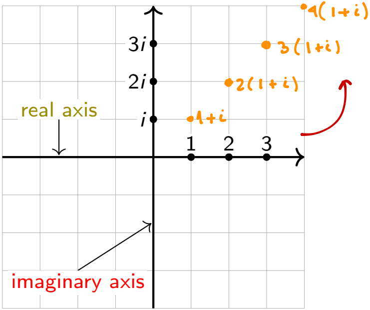
Addition: Same as vector addition $(a+bi)+(c+di)=(a+b)+(b+d)i$.

Usually, *vectors cannot be multiplied*, but complex numbers can!

Multiplication:

$$(a+bi)(c+di) = ac + (bc+ad)i + bdi^2 = (ac-bd)+(bc+ad)i.$$

A plane representation of **multiplication** of $\{1,2,3,\ldots\}$ by complex $z=1+i$



When z is a *real* number, multiplication *means stretching*.

When z has an imaginary part, **multiplication also means rotation**.

Operations on Complex Numbers (II)

The conjugate

For a complex number $z = a + bi$, the **complex conjugate** of z is $\overline{z} = a - bi$.

The following is a convenient definition because:

- If $z = a + bi$ then

$$z\overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$$

  which is a real number!

- Note that the *length* of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\sqrt{a^2 + b^2}$,
- There is no geometric interpretation of complex division, but if $z \neq 0$ then:

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}.$$

Example:

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{1^2 + (-1)^2} = \frac{1 + 2i + i^2}{2} = i.$$

Notation and Polar coordinates

Real and imaginary part: $\operatorname{Re}(a+bi)=a \qquad \operatorname{Im}(a+bi)=b$.

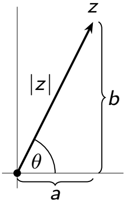
Absolute value: $|a+bi|=\sqrt{a^2+b^2}$.

Some properties

$$\overline{z+w}=\overline{z}+\overline{w} \qquad |z|=\sqrt{z\overline{z}}$$
$$\overline{zw}=\overline{z}\cdot\overline{w} \qquad |zw|=|z|\cdot|w|$$

Any complex number $z=a+bi$ has the **polar coordinates**: *angle and length*.

- The length is $|z|=\sqrt{a^2+b^2}$
- The angle $\theta=\arctan(b/a)$ is called the **argument** of z, and is denoted $\theta=\arg(z)$.



The relation with cartesian coordiantes is:

$$z=|z|\underbrace{(\cos\theta+i\sin\theta)}_{\text{unit `vector'}}.$$

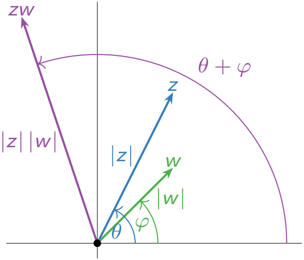
More on multiplication

It turns out that multiplication has a **precise geometric meaning**:

Complex multiplication

Multiply the absolute values and add the arguments:

$$|zw| = |z|\,|w| \qquad \arg(zw) = \arg(z) + \arg(w).$$



- Note $\arg(\overline{z}) = -\arg(z)$.
- Multiplying z by $\overline{z}$ gives a real number because the *angles cancel out*.

The point of using complex numbers is to find all eigenvalues of the characteristic polynomial.

Fundamental Theorem of Algebra

Every **polynomial** of degree n has *exactly n complex roots*, counted *with multiplicity*. That is, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for (*not necessarily distinct*) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Conjugate pairs of roots

If f is a polynomial with **real coefficients**, then the *complex roots* of real polynomials come in *conjugate pairs*. (Real roots are conjugate of themselves).

Degree 2:

The quadratic formula gives the (real or complex) roots:

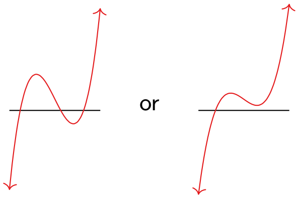
$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For **real polynomials**, the roots are *complex conjugates if $b^2 - 4c$ is negative*.

Degree 3:

A **real cubic polynomial** has either three real roots, or one real root and a conjugate pair of complex roots.

The graph looks like:



respectively.

Example Degree 2:

If $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Example Degree 3:

Let $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$.

Since $f(2) = 0$, we can do polynomial long division by $\lambda - 2$:
We get $f(\lambda) = (\lambda - 2)\left(5\lambda^2 - 8\lambda + 5\right)$. Using the quadratic formula, the second polynomial has a root when

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{10} = \frac{4}{5} \pm \frac{\sqrt{-36}}{10} = \frac{4 \pm 3i}{5}.$$

Therefore,

$$f(\lambda) = 5(\lambda - 2)\left(\lambda - \frac{4 + 3i}{5}\right)\left(\lambda - \frac{4 - 3i}{5}\right).$$

Poll

The characteristic polynomial of

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$. This has two complex roots $(1 \pm i)/\sqrt{2}$.

Poll

Let's allow vectors with complex entries. What are the *eigenvectors of A*?

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$
$$\text{the eigenvalue } \frac{1-i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Do you **notice a pattern?**

Conjugate Eigenvectors

Allowing complex numbers both eigenvalues and eigenvectors of **real square matrices** occur in conjugate pairs.

Conjugate eigenvectors

Let A be a **real square matrix**. If λ is an eigenvalue with eigenvector v, *then* $\overline{\lambda}$ is an eigenvalue with *eigenvector* $\overline{v}$.

Conjugate pairs of roots in polynomial:

If λ is a root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right). \end{aligned}$$

Conjugate pairs of eigenvectors:

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}.$$

Classification of 2×2 Matrices with no Real Eigenvalue

Triptych

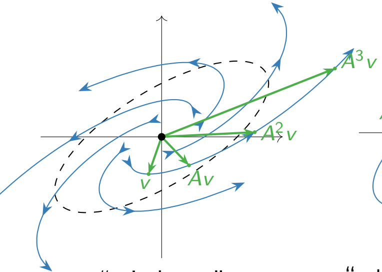
Pictures of *sequence of vectors $v, Av, A^2v, \ldots$* $M = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$

(in general, a real matrix with not real eigenvalues, depending on the length of eigenvalues).

$$A = \sqrt{2}M$$

$$\lambda = \frac{\sqrt{3}-i}{\sqrt{2}}$$

$|\lambda| > 1$

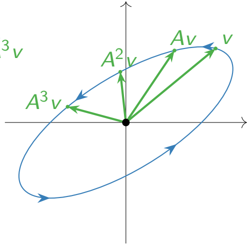


"spirals out"

$$A = M$$

$$\lambda = \frac{\sqrt{3}-i}{2}$$

$|\lambda| = 1$

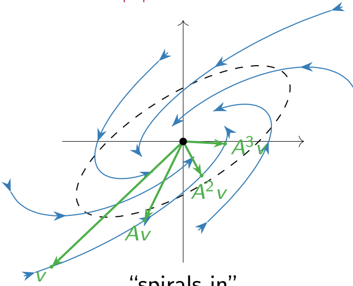


"rotates around an ellipse"

$$A = \frac{1}{\sqrt{2}}M$$

$$\lambda = \frac{\sqrt{3}-i}{2\sqrt{2}}$$

$|\lambda| < 1$



"spirals in"

Picture with 2 Real Eigenvalues

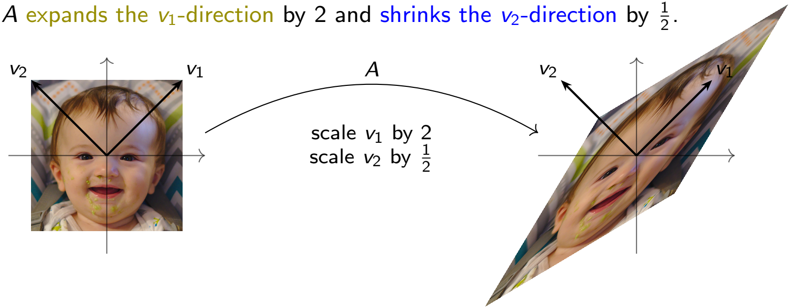
Recall the pictures for a matrix with 2 real eigenvalues.

Example: Let $A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

So A expands the v_1-direction by 2 and shrinks the v_2-direction by $\frac{1}{2}$.



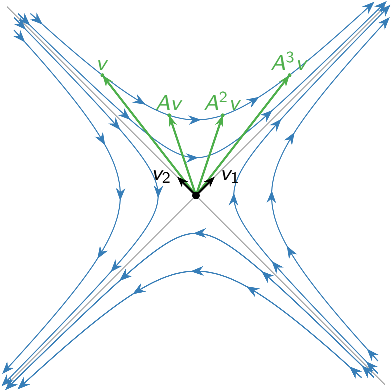
Picture with 2 Real Eigenvalues

We can also draw the *sequence of vectors $v, Av, A^2v, \ldots$*

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\lambda_1 = 2 \qquad \lambda_2 = \tfrac{1}{2}$$

$$|\lambda_1| > 1 \qquad |\lambda_1| < 1$$



Exercise:

Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.