Announcements

Monday, November 13

- Midterm 3 this Friday
  - Exam covers: Sections 3.1,3.2,5.1,5.2,5.3 and 5.5
  - Many problems will be computational (see Practice exam, e.g. Problem 4)
  - For the rest 20-30% you need to have understood the motivation/high-level idea of the topics
- Review: **Recitation Style at Howey L4 Wednesday 5-6pm**
  - Solve and discuss Practice problems in groups
  - Preparing for the exam tips and strategies
  - It is not mandatory

Section 5.5

Complex Eigenvalues (Part II)

Motivation: Complex Versus Two Real Eigenvalues

Today's decomposition is very analogous to diagonalization.

Theorem

Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated *eigenvalues* λ_1, λ_2. Then

$$A = PDP^{-1}$$

where

scale x-axis by λ_1
scale y-axis by λ_2

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Theorem

Let A be a 2×2 **real matrix** with a complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \mathrm{Re}\, v & \mathrm{Im}\, v \\ | & | \end{pmatrix} \quad \text{and} \quad C = (\text{rotation})\cdot(\text{scaling}).$$

Computing Eigenvectors of 2×2 Matrices

Specially useful for complex eigenvalues

Let A be a 2×2 matrix, and let λ be an eigenvalue of A.

Then $A - \lambda I$ is not invertible, so its rows are *linearly independent*.

When we row reduce, the second row entries are zeros.

Save time! there is *no need to find* the exact entries of *second row*

2×2 Shortcut

If $A - \lambda I = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix}$, then use $\begin{pmatrix} b \\ -a \end{pmatrix}$ or $\begin{pmatrix} -b \\ a \end{pmatrix}$ *as eigenvector* for λ.

Example:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = \frac{1-i}{\sqrt{2}}.$$

Then:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix}$$

so an eigenvector is $v = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Poll Example Completed

Last poll used the matrix of **rotation by $\pi/4$**:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{has eigenvalues} \qquad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Compute *an eigenvector* for $\lambda = (1+i)/\sqrt{2}$ (factor out $\sqrt{2}$):

$$A - \lambda I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

Row reducing:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $-ix = y$, so an eigenvector is $v = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

When row reducing *we can also divide by i* and obtain $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Can use a similar computation for conjugate eigenvalue. Instead, *save time:* take conjugates for both λ and v above.

A 3×3 Example

Find the complex eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

This factors out automatically if you expand cofactors along the third row or column

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)} \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4+3i}{5}, \quad \frac{4-3i}{5}.$$

Recall: When we *find an eigenvector v* with eigenvalue λ then we automatically know that *$\overline{v}$ is eigenvector* with eigenvalue $\overline{\lambda}$.

A 3×3 Example
Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Find eigenvector v with eigenvalue $\frac{4+3i}{5}$. Row reduce:

$$A - \frac{4+3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i\text{,swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so *an eigenvector is* $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

Therefore, an *eigenvector with conjugate eigenvalue* $\dfrac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Theorem

Let A be a 2×2 **real matrix** with a *complex (non-real)* eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

The matrix C is a *composition of scaling* by $|\lambda|$ *and rotation* by $\theta = -\arg(\lambda)$:

$$C = |\lambda| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

With a complex eigenvalue λ

The matrix C correspond to multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

The matrix A is similar to C; that is to a **rotation by the argument of $\overline{\lambda}$** composed with scaling by $|\lambda|$.

What does $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is $f(\lambda) = \lambda^2 - 2\lambda + 2$.

  By the quadratic formula, the roots are $\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$.
- For $\lambda = 1 - i$, we compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

- Therefore, $A = PCP^{-1}$ where

$$P = \left(\mathrm{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \mathrm{Im}\begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

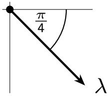
Decomposition: Geometric Interpretation

Example 1, continued

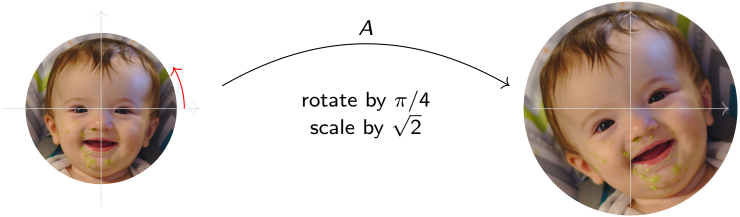
- The matrix $C = A$ *scales by a factor* of

$$|\lambda| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

- The argument of λ is $-\pi/4$:



Therefore $C = A$ *rotates by* $+\pi/4$.



What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is $f(\lambda) = \lambda^2 - 2\sqrt{3}\,\lambda + 4$. By the quadratic formula, the roots are $\frac{2\sqrt{3}\pm\sqrt{12-16}}{2} = \sqrt{3} \pm i$.
- For $\lambda = \sqrt{3} - i$, we compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} 1+i & -2 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

- It follows that $A = PCP^{-1}$ where

$$P = \left(\operatorname{Re}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \operatorname{Im}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

Decomposition: Geometric Interpretation

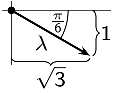
Example 2, continued

$$A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad C = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \qquad \lambda = \sqrt{3} - i$$

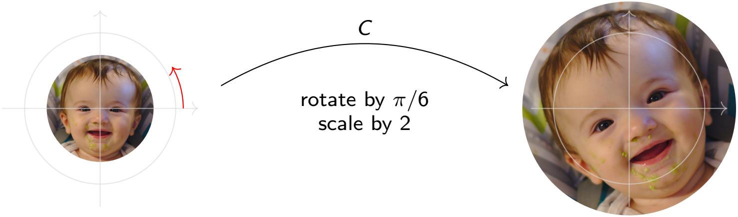
- The matrix C scales by a factor of

$$|\lambda| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$$

- The argument of λ is $-\pi/6$:

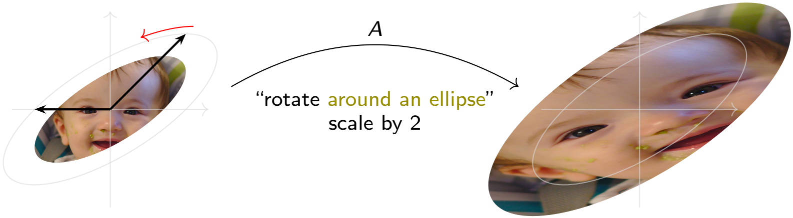


Therefore C *rotates by* $+\pi/6$.



Decomposition: Geometric Interpretation
Example 2, continued

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?



$A = PCP^{-1}$ does the same as C.

but with *respect to the basis* $\mathcal{P} = \left\{ \binom{1}{1}, \binom{-1}{0} \right\}$ of columns of P

The 3-Dimensional Case

Theorem

Let A be a **real 3×3 matrix**. Suppose that A has

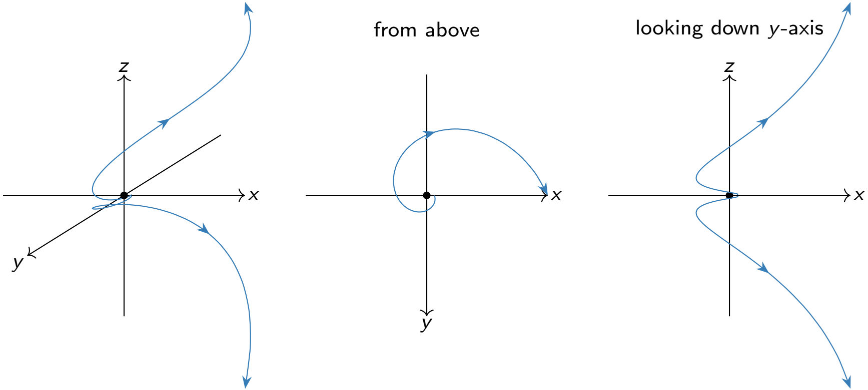
- one real eigenvalue λ_1 with eigenvector v_1,
- and one conjugate pair of complex eigenvalues $\lambda_2, \overline{\lambda}_2$ with eigenvectors $v_2, \overline{v}_2$.

Then $A = PCP^{-1}$, where

$$P = \begin{pmatrix} | & | & | \\ v_1 & \operatorname{Re} v_2 & \operatorname{Im} v_2 \\ | & | & | \end{pmatrix} \quad C = \begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \operatorname{Re} \lambda_2 & \operatorname{Im} \lambda_2 \\ 0 & -\operatorname{Im} \lambda_2 & \operatorname{Re} \lambda_2 \end{pmatrix}$$

1. C is a **block diagonal**
2. The columns of P form a basis for the eigenspace for the real eigenvector, and have columns $(\operatorname{Re} v \;\; \operatorname{Im} v)$ for the pair of non-real eigenvectors.
3. The *order of blocks* in C determines the *order of columns* in P.

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This acts *on the xy-plane by rotation* by $\pi/4$ and scaling by $\sqrt{2}$. This acts *on the z-axis by scaling* by 2. Pictures:



Note: A is already a block diagonal. In general, this *dynamics occur along the axes* given by the columns of P (if $A = PCP^{-1}$).

Extra: The n-Dimensional Case

Theorem

Let A be a **real $n \times n$ matrix**. Suppose that for each (real or complex) eigenvalue, the *dimension of the eigenspace equals the algebraic multiplicity*.

Then $A = PCP^{-1}$, where

1. C is **block diagonal**:
   - the blocks containing the *real eigenvalues* (with their multiplicities) are 1×1 blocks.
   - the blocks containing the pairs of conjugate *complex eigenvalues* (with their multiplicities) are 2×2 blocks.

     $$\begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}$$

     (λ must have an imaginary part)
2. P has columns that either *form bases for the eigenspaces* for the real eigenvectors, or *come in pairs* $\begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix}$ for the non-real eigenvectors.

Extra: Why This Is Not A Weird Thing To Do

An anachronistic historical aside

In the beginning, people only used *counting numbers for, well, counting things*: $1, 2, 3, 4, 5, \ldots$.
Then someone (Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī, 825) had the ridiculous idea that there *should be a number that represents an absence of quantity (number 0)*. This blew everyone's mind.

Then it occurred to someone (Chinese mathematician Liu Hui, c. 3rd century) that there should be *negative numbers to represent a deficit in quantity*. That seemed reasonable, until people realized that $10 + (-3)$ would have to equal 7. This is when people started saying, "bah, math is just too hard for me."

At this point *it was inconvenient that you couldn't divide 2 by 3*. Thus someone (Indian mathematician Aryabhatta, c. 5th century) invented fractions *(rational numbers) to represent fractional quantities*. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans (c. 6th century BCE) discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The *real number $\sqrt{2}$, which is not a fraction*, was thus invented to solve the equation $x^2 - 2 = 0$.

Now we come to invent a *number i* that *solves the equation $x^2 + 1 = 0$*.