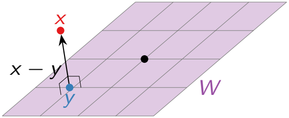
Section 6.2

Orthogonal Sets

Best Approximation

Due to measurement error, the measured x is not actually in the subspace it must lie on (*for theoretical reasons*).



Best approximation: y is the *closest point* to x on W.

Replace x with its orthogonal projection y onto W.

How do you know that y is the closest point?

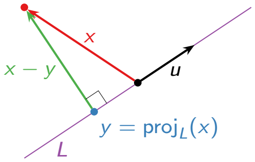
Orthogonal Projection onto a Line

Theorem

Let $L = \mathrm{Span}\{u\}$ *be a line* in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The closest point to x on L is the point

$$\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$

This point is called the **orthogonal projection of x onto L.**



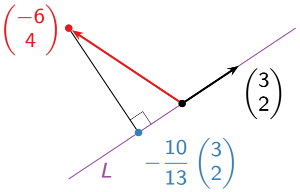
Choose term 'ortogonal' because *$x - y$ is in $L^{\perp}$*.

Orthogonal Projection onto a Line

Example

Compute the *orthogonal projection* of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line *L spanned by* $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$$



Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. Such set is **orthonormal** if, in addition, each vector is a *unit vector*.

Example: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is an orthogonal set.

Check:

Orthogonal bases

Linearly independent

An *orthogonal set of vectors* $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ is linearly independent. Therefore $\mathcal{B}$ **forms a basis** for $W = \operatorname{Span} \mathcal{B}$.

$\mathcal{B}$-coordinates for Orthogonal bases

Theorem

Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span} \mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

An advantage

For orthogonal bases, is it's *easy to compute the $\mathcal{B}$-coordinates* of a vector x in W:

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

Orthogonal Bases

Geometric reason

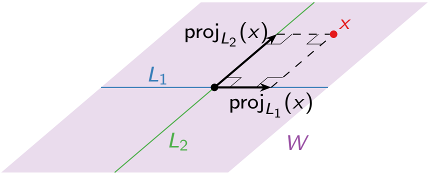
Theorem

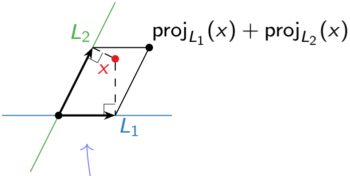
Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span}\mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \overset{\text{proj}_{L_2}(u_2)}{\boxed{\frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2}} + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

If L_i is the line spanned by u_i, then this says

$$x = \text{proj}_{L_1}(x) + \text{proj}_{L_2}(x) + \cdots + \text{proj}_{L_m}(x).$$





Warning: This only works for an *orthogonal* basis.

Orthogonal Bases

Example

Problem: Find the $\mathcal{B}$-coordinates of $x = \binom{0}{3}$, where

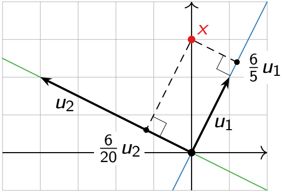
$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}.$$

Old way:

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \xrightarrow{\text{rref}} \qquad\qquad .$$

New way: Exploit that $\mathcal{B}$ is an *orthogonal basis*.

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2$$



$$\Longrightarrow \quad [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

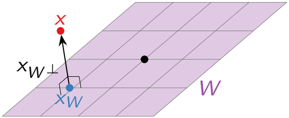
Orthogonal Bases

Example

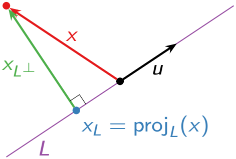
Section 6.3

Orthogonal Projections

Motivation



Example with a line: The closest point to x in L is $\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$



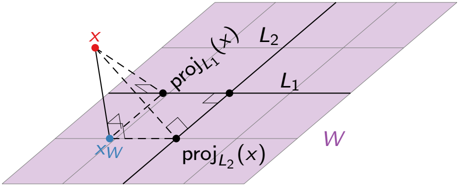
Orthogonal Projections

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

The orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



Orthogonal Projections

Best approximation

Every vector x can be *decompsed uniquely* as $x = x_W + x_{W^\perp}$ where

- $x_W = y$ is the *closest vector* to x *in* W, and
- $x_{W^\perp} = x - y$ is in $W^\perp$.

Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $\mathrm{proj}_W(x)$ is the closest point to x in W. Therefore

$$\boxed{x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x)}.$$

We can think of orthogonal projection as a *transformation*:

$$\mathrm{proj}_W \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \mathrm{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\mathrm{proj}_W(x) = x$.
3. For every x in $W^\perp$, we have $\mathrm{proj}_W(x) = 0$.
4. The range of proj_W is W.

The following is *the property we wanted* all along.

Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $y = \mathrm{proj}_W(x)$ is *the closest point in W to x*, in the sense that

$$\mathrm{dist}(x, y) \leq \mathrm{dist}(x, y') \quad \text{for all} \quad y' \text{ in } W.$$

Poll

What is the matrix for $\text{proj}_W \colon \mathbf{R}^3 \to \mathbf{R}^3$, where

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Answer: Recall how to compute the matrix for a linear transformation:

$$A = \begin{pmatrix} | & | & | \\ \text{proj}_W(e_1) & \text{proj}_W(e_2) & \text{proj}_W(e_3) \\ | & | & | \end{pmatrix}.$$

We compute:

Let A **be the matrix for** proj_W, where W is an m-dimensional subspace of $\mathbf{R}^n$.

Facts:

1. A *is diagonalizable* with eigenvalues 0 and 1;
2. it is similar to the *diagonal matrix with m ones and $n - m$ zeros* on the diagonal, and
3. $A^2 = A$.

Example: If W is a plane in $\mathbf{R}^3$, then A is similar to projection onto the xy-plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

What is the (minimum) *distance from e_1 to W* $=$ Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

Answer: From e_1 to its closest point on W:

$$\mathrm{dist}(e_1, \mathrm{proj}_W(e_1)) = \|(e_1)_{W^\perp}\|.$$