Review from Chapter 6

Selected Topics

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

To go from an orthogonal set $\{u_1, u_2, \ldots, u_m\}$ to an orthonormal set, replace each u_i with $u_i / \|u_i\|$.

Theorem

An orthogonal set is linearly independent. In particular, it is a basis for its span.

Orthogonal Projection

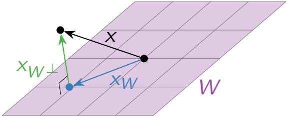
Let W be a subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

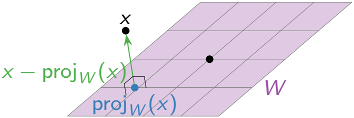
$$\operatorname{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

This is the closest vector to x that lies on W. In other words, the difference $x - \operatorname{proj}_W(x)$ is perpendicular to W: it is in $W^\perp$. Notation:

$$\boxed{x_W = \operatorname{proj}_W(x) \qquad x_{W^\perp} = x - \operatorname{proj}_W(x).}$$

So x_W is in W, $x_{W^\perp}$ is in $W^\perp$, and $x = x_W + x_{W^\perp}$.





Orthogonal Projection

Special cases

Special case: If x is in W, then $x = \text{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

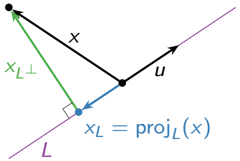
In other words, the $\mathcal{B}$-coordinates of x are

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_1 \cdot u_2}, \ldots, \frac{x \cdot u_m}{u_1 \cdot u_m} \right),$$

where $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$, an orthogonal basis for W.

Special case: If $W = L$ is a line, then $L = \text{Span}\{u\}$ for some nonzero vector u, and

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u$$



Orthogonal Projection

And matrices

Let W be a subspace of $\mathbf{R}^n$.

Theorem

The orthogonal projection proj_W is a *linear* transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$. Its range is W.

If A is the matrix for proj_W, then $A^2 = A$ because projecting twice is the same as projecting once: $\text{proj}_W \circ \text{proj}_W = \text{proj}_W$.

Theorem

The only eigenvalues of A are 1 and 0.

Why?

$$Av = \lambda v \implies A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v.$$

So if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2. But $A^2 = A$, so $\lambda^2 = \lambda$, and hence $\lambda = 0$ or 1.

The 1-eigenspace of A is W, and the 0-eigenspace is $W^\perp$.

The Gram–Schmidt Process

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis for a subspace W of $\mathbf{R}^n$. Define:

1. $u_1 = v_1$
2. $u_2 = v_2 - \mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) \qquad = v_2 - \dfrac{v_2 \cdot u_1}{u_1 \cdot u_1}\, u_1$
3. $u_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2\}}(v_3) \qquad = v_3 - \dfrac{v_3 \cdot u_1}{u_1 \cdot u_1}\, u_1 - \dfrac{v_3 \cdot u_2}{u_2 \cdot u_2}\, u_2$

$\vdots$

m. $u_m = v_m - \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{m-1}\}}(v_m) \; = v_m - \displaystyle\sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i}\, u_i$

Then $\{u_1, u_2, \ldots, u_m\}$ is an *orthogonal* basis for the same subspace W.

In fact, for each i,

$$\mathrm{Span}\{u_1, u_2, \ldots, u_i\} = \mathrm{Span}\{v_1, v_2, \ldots, v_i\}.$$

Note if v_i is in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\} = \mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}$, then $v_i = \mathrm{proj}_{\mathrm{Span}\{u_1, u_2, \ldots, u_{i-1}\}}(v_i)$, so $u_i = 0$. So this also detects linear dependence.

Review: Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\, A = \mathrm{Span}\{\text{columns of } A\}$.
- The range of a linear transformation (same as above).
- The *null space* of a matrix: $\mathrm{Nul}\, A = \{x \mid Ax = 0\}$.
- The *row space* of a matrix: $\mathrm{Row}\, A = \mathrm{Span}\{\text{rows of } A\}$.
- The λ-eigenspace of a matrix, where λ is an eigenvalue.
- The orthogonal complement $W^{\perp}$ of a subspace W.
- The zero subspace $\{0\}$.
- All of $\mathbf{R}^n$.

Review: Subspaces and Bases

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- Can you write V as one of the examples on the previous slide?
- If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of $\mathbf{R}^n$ is any collection of vectors whatsoever. Like, the unit circle in $\mathbf{R}^2$, or all vectors with whole-number coefficients. A *subspace* is a subset that satisfies three additional properties. Most subsets are not subspaces.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.
2. Similarity has nothing to do with row equivalence.
3. Similar matrices usually do not have the same eigenvectors.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue $1/2$.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{\binom{1}{1}, \binom{1}{-1}\right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

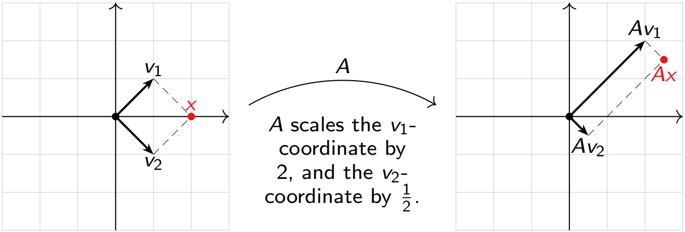
To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.
3. Compute y from $[y]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.
3. $y = 2v_1 + \frac{1}{2}v_2 = \binom{5/2}{3/2}$.

Picture:



Review: Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns $v_1, v_2, \ldots, v_n$, then

$$b = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

So if $Ax = b$ has a solution, then b is a linear combination of $v_1, v_2, \ldots, v_n$, and conversely. Equivalently, b is in $\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = \mathrm{Col}\, A$.

Important

$Ax = b$ is consistent if and only if b is in $\mathrm{Col}\, A$.

Least-Squares Solutions

Suppose that $Ax = b$ is *in*consistent. Let $\widehat{b} = \text{proj}_{\text{Col } A}(b)$ be the closest vector for which $A\widehat{x} = \widehat{b}$ *does* have a solution.

Definition

A solution to $A\widehat{x} = \widehat{b}$ is a **least squares solution** to $Ax = b$. This is the solution $\widehat{x}$ for which $A\widehat{x}$ is *closest* to b (with respect to the usual notion of distance in $\mathbf{R}^n$).

Theorem

The least-squares solutions to $Ax = b$ are the solutions to

$$A^T A\widehat{x} = A^T b.$$

If A has *orthogonal* columns $u_1, u_2, \ldots, u_n$, then the least-squares solution is

$$\widehat{x} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \; \frac{x \cdot u_2}{u_2 \cdot u_2}, \; \cdots, \; \frac{x \cdot u_m}{u_m \cdot u_m} \right)$$

because

$$A\widehat{x} = \widehat{b} = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$