Math 1553 Worksheet §1.3

Solutions

1. Is it possible to write

$$b = \begin{pmatrix} -3 \\ -9 \\ 7 \end{pmatrix} \text{ as a linear combination of } \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix}?$$

If your answer is no, justify why not. If your answer is yes, write b as a linear combination of those four vectors.

Solution.

We are trying to find scalars x_1 through x_4 so that

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ 7 \end{pmatrix}.$$

In other words, we are trying to solve

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= -3 \\ 2x_1 + 3x_2 + x_3 - 5x_4 &= -9 \\ x_1 + 3x_2 - x_3 - 6x_4 &= 7 \end{aligned} \quad \rightsquigarrow \quad (x_1, x_2, x_3, x_4) = (?, ?, ?, ?).$$

First we translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -3 \\ 2 & 3 & 1 & -5 & -9 \\ 1 & 3 & -1 & -6 & 7 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -32 \\ 0 & 1 & -1 & 0 & 45 \\ 0 & 0 & 0 & 1 & 16 \end{array}\right)$$

This translates back to the system of equations

$$\begin{aligned} x_1 \qquad + 2x_3 \qquad &= -32 \\ x_2 - x_3 \qquad &= 45 \\ x_4 &= 16. \end{aligned}$$

The rightmost column is not a pivot column, so the system is consistent. The only free variable is x_3; moving it to the right side of the equation gives the parametric form

$$x_1 = -32 - 2x_3 \qquad x_2 = 45 + x_3 \qquad x_3 \text{ is free} \qquad x_4 = 16.$$

Thus, there are infinitely many ways to write b as a linear combination of the four vectors given in the problem, depending on what you choose x_3. For example, when $x_3 = 0$, we get $x_1 = -32,\ x_2 = 45,\ x_4 = 16$, so

$$-32 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 45 \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 16 \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \\ 7 \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Is b in the span of the columns of A? Justify your answer.

Solution.

Let v_1, v_2, and v_3 be the columns of A. To say b is in the span of the columns of A is to say that $b = x_1v_1 + x_2v_2 + x_3v_3$ for some scalars x_1, x_2, and x_3, which means

$$\begin{aligned} x_1 \qquad\;\; + 5x_3 &= 2 \\ -2x_1 + \;\; x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= 6 \end{aligned}$$

We translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent. Therefore, b is in the span of the columns of A. In fact, we can take $x_1 = 2, x_2 = 3$, and $x_3 = 0$, to write

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

3. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

b) The span of any set contains the zero vector.

Solution.

a) This is **false.** For instance, the vectors

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}\right\}$$

only span the x-axis.

b) This is **true.** We have

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector v, you get $v+$(no other summands), which is just v; and the only vector which gives you v when you add it to v, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)

4. Zander has challenged you to find his hidden treasure, located at some point (a, b, c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Zander's message, you have discovered that the treasure's first and second entries are (in order) −4 and 3.

a) What is the treasure's full location?

b) Give instructions for how to find the treasure by only moving in the directions given by v_1, v_2, and v_3.

Solution.

a) We translate this problem into linear algebra. Let c be the final entry of the treasure. Since Zander has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2, and v_3 (in other words, find c so that the treasure's location is in in $\text{Span}\{v_1, v_2, v_3\}$). We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$, or $c = 5$.

The location of the treasure is $(-4, 3, 5)$.

b) Getting to the point $(-4, 3, 5)$ using the vectors v_1, v_2, and v_3 is equivalent to finding scalars x_1, x_2, and x_3 so that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4 \\ -x_1 - 4x_2 + x_3 &= 3 \\ -2x_1 - 7x_2 \phantom{{}+3x_3} &= 5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note x_3 is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$