**Math 1553 Worksheet §5.1, 5.2**

**1.** True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that $A$ is an $n \times n$ matrix.

**a)** If $v_1$ and $v_2$ are linearly independent eigenvectors of $A$, then they must correspond to different eigenvalues.

**b)** The entries on the main diagonal of $A$ are the eigenvalues of $A$.

**c)** The number $\lambda$ is an eigenvalue of $A$ if and only if there is a nonzero solution to the equation $(A - \lambda I)x = 0$.

**d)** To find the eigenvectors of $A$, we reduce the matrix $A$ to row echelon form.

**e)** If $A$ is invertible and 2 is an eigenvalue of $A$, then $\frac{1}{2}$ is an eigenvalue of $A^{-1}$.

## Solution.

**a)** False. For example, if $A = I_2$ then $e_1$ and $e_2$ are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

**b)** False. This is true if $A$ is triangular, but not in general.
For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

**c)** True.

$$(A - \lambda I)x = 0 \iff Ax - \lambda x = 0 \iff Ax = \lambda x.$$

Therefore, $(A - \lambda I)x = 0$ has a nonzero solution if and only if $Ax = \lambda x$ has a nonzero solution, which is to say that $\lambda$ is an eigenvalue of $A$.

**d)** False. The RREF of $A$ gives us almost no info about eigenvalues or eigenvectors. To get the eigenvectors corresponding to an eigenvalue $\lambda$, we put $A - \lambda I$ into RREF and write the solutions of $\left(A - \lambda I \mid 0\right)$ in parametric vector form.

**e)** True. Let $v$ be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, $v$ is an eigenvector of $A^{-1}$ corresponding to the eigenvalue $\frac{1}{2}$.

**2.** In what follows, $T$ is a linear transformation with matrix $A$. Find the eigenvectors and eigenvalues of $A$ without doing any matrix calculations. (Draw a picture!)

**a)** $T =$ identity transformation of $\mathbf{R}^3$.

**b)** $T =$ projection onto the $xz$-plane in $\mathbf{R}^3$.

**c)** $T =$ reflection over $y = 2x$ in $\mathbf{R}^2$.

## Solution.

**a)** $T(x) = x$ for all $x$, so every nonzero vector in $\mathbf{R}^3$ is an eigenvector for $T$ corresponding to eigenvalue $\lambda = 1$.

**b)** $T(x, y, z) = (x, 0, z)$, so $T$ fixes every vector in the $xz$-plane and destroys every vector of the form $(0, a, 0)$ with $a$ real. Therefore, $\lambda = 1$ and $\lambda = 0$ are eigenvalues and in fact they are the only eigenvalues since their combined eigenvectors span all of $\mathbf{R}^3$:

The eigenvectors for $\lambda = 1$ are all vectors of the form $\begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ where at least one of $x$ and $z$ is nonzero, and the eigenvectors for $\lambda = 0$ are all vectors of the form $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ where $y \neq 0$.

**c)** $T$ fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

$T$ flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2, 1) = (2, -1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

**3.** Let $A = \begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$. Find the eigenvalues of $A$.

## Solution.

We find the characteristic polynomial $\det(A - \lambda I)$ any way we like. The computation below uses the cofactor expansion along the second row:

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{bmatrix} = (1-\lambda) \cdot \det \begin{bmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{bmatrix}$$

$$= (1-\lambda) \cdot [(5-\lambda)(-2-\lambda) - 3 \cdot 6] = (1-\lambda)(\lambda^2 - 3\lambda - 28)$$

$$= -\lambda^3 + 4\lambda^2 + 25\lambda - 28 \quad \text{or} \quad (1-\lambda)(\lambda-7)(\lambda+4)$$

The characteristic equation is thus $(1-\lambda)(\lambda-7)(\lambda+4) = 0$, so the eigenvalues are $\lambda = -4$, $\lambda = 1$, and $\lambda = 7$.

**Supplemental Problems**

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

**1.** Find a basis $\mathcal{B}$ for the $(-1)$-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

## Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0\right) = \left(Z + I \mid 0\right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue $-1$:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

**2.** Suppose $A$ is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of $A$. Justify your answer.

## Solution.

If $\lambda$ is an eigenvalue of $A$ and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of $A$.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of $A$. Therefore, the only eigenvalue of $A$ is 0.

**3.** Give an example of matrices $A$ and $B$ which have the same eigenvalues and the same algebraic multiplicities for each eigenvalue, but which are *not* similar. Justify why they are not similar.

## Solution.

Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Both $A$ and $B$ have characteristic equation $\lambda^2 = 0$, so each has eigenvalue $\lambda = 0$ with algebraic multiplicity two. However, the only matrix similar to $A$ is the zero matrix: if $P$ is any invertible $2 \times 2$ matrix then $P^{-1}AP = P^{-1}0P = 0$. Therefore, $A$ and $B$ are not similar.

**4.** Using facts about determinants, justify the following fact: if $A$ is an $n \times n$ matrix, then $A$ and $A^T$ have the same characteristic polynomial.

## Solution.

We will use three facts which apply to all $n \times n$ matrices $B,\ Y,\ Z$:

(1) $\det(B) = \det(B^T)$.

(2) $(Y - Z)^T = Y^T - Z^T$

(3) If $\lambda$ is any scalar then $(\lambda I)^T = \lambda I$ since the identity matrix is completely symmetric about its diagonal.

Using these three facts in order, we find

$$\det(A - \lambda I) = \det\left((A - \lambda I)^T\right) = \det\left(A^T - (\lambda I)^T\right) = \det(A^T - \lambda I).$$