MATH 1553
MIDTERM EXAMINATION 1

Name		Section	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculators, etc.) allowed.
- Read carefully every problem before working on the answers.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

Circle **T** if the statement is always true and circle **F** if the statement is ever false. Do not assume any information that is not stated.

a) **T** **F** [2pts] Two vectors v_1, v_2 in $\mathbf{R}^3$ always span a plane.

b) **T** **F** [2pts] Let A be an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$. The equation $Ax = b$ is homogeneous if the zero vector is a solution.

c) **T** **F** [2pts] The solution set of $Ax = b$ is parallel to the solution set of $Ax = 0$.

d) **T** **F** [2pts] The following augment matrix corresponds to a system with a unique solution:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

e) [2pts] (Unrelated) Provide an example of a matrix in row echelon form that is not in reduced row echelon form. Explain which condition of reduced echelon forms is missing.

Solution.

a) False: For example, $v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ span only a line.

b) True: If the zero vector is a solution, that is the vector $x = (0, \ldots, 0) \in \mathbf{R}^m$, then the prodcut Ax results in the zero vector as well.

c) False. It may be that the equation $Ax = b$ is inconsistent, so that the solution set is empty.

d) False: The augmented matrix has a non-pivot column, therefore there are infinitely many solutions.

e) For example, the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is in row echelon from. However the second column has a pivot but it has more that one non-zero entry. Also, the pivot in the last column is not equal to 1.

Problem 2.

Consider the following system of equations:

$$\begin{aligned} -2x_1 + 2x_2 + 4x_3 &= 6 \\ -2x_2 + x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= 7 \end{aligned}$$

a) [1pts] Write the above system as an augmented matrix.

b) [2pts] Write the above system as a vector equation.

c) [2pts] Write the above system as a matrix equation.

d) [5pts] Is the system consistent? Justify your answer.

Solution.

a)

$$\left(\begin{array}{rrr|r} -2 & 2 & 4 & 6 \\ 0 & -2 & 1 & 2 \\ 3 & -1 & 2 & 7 \end{array}\right)$$

b)

$$x_1 \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$

c)

$$\begin{pmatrix} -2 & 2 & 4 \\ 0 & -2 & 1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix}$$

d) Yes, it is consistent. By row reduction we can find that

$$\begin{pmatrix} -2 & 2 & 4 \\ 0 & -2 & 1 \\ 3 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & -2 & 1 \\ 3 & -1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & -2 & 1 \\ 0 & 2 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 9 \end{pmatrix}$$

(For this question, there is no need to row reduce further).

Once the matrix is reduced to an echelon form we can verify that there is a pivot in each row. Therefore, the system $Ax = b$ is consistent for any b.

Problem 3.

Consider the matrix equation $Ax = 0$, where

$$A = \begin{pmatrix} 2 & 4 & -2 & 0 \\ -2 & -4 & 0 & 2 \\ 1 & 2 & 2 & -3 \end{pmatrix}.$$

a) [7pts] Give the parametrized vector solution of the equation above. How many free variables there are?

b) [3pts] What best describes the geometry of the solutions to $Ax = 0$ above? Select all that apply.
(1) Solution set is a line.
(2) Solution set is a plane.
(3) Solution set does not pass through the origin.

Solution.

a) First we row reduced the matrix A

$$\begin{pmatrix} 2 & 4 & -2 & 0 \\ -2 & -4 & 0 & 2 \\ 1 & 2 & 2 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 4 & -2 & 0 \\ 0 & 0 & -2 & 2 \\ 1 & 2 & 2 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 2 & -3 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, the second and fourth columns in the matrix have no pivot. Therefore, the system corresponding to $Ax = 0$ has 2 free variables; namely, x_2 and x_4.

The system is equivalent to

$$\begin{array}{rcr} x_1 + 2x_2 & & -x_4 = 0 \\ + & & -x_3 + x_4 = 0 \end{array} \quad \text{or} \quad \begin{array}{rcrcr} x_1 & = & -2x_2 & + & x_4 \\ x_2 & = & +x_2 & & \\ x_3 & = & & & +x_4 \\ x_4 & = & & & +x_4 \end{array}$$

and in parametric vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

b) The solution set can be written using two free variables, therefore it is a plane.

It passes through the origin because $A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, so the zero vector is a solution (or simply, because the system is homogeneous).

Problem 4.

A company manufactures frisbees and boomerangs. For each frisbee, the company spends \$.25 on materials, \$.25 on labor and \$.10 on overhead. For each boomerang, the company spends \$.25 on materials, \$.50 on labor and \$.20 on overhead. Finally, the company has a total budget of \$1000 on materials, \$1400 on labor and \$560 on overhead.

a) [3pts] If the company produces x_1 frisbees and x_2 boomerangs, describe the various costs the company has.

b) [5pts] Is there a combination of production that will require exactly all the budget? If yes, give all possible values of x_1, x_2.

c) [2pts] Describe a vector equation that models the question in **b)**

Solution.

a) The costs per unit frisbee and boomerang are represented by $\begin{pmatrix} .25 \\ .25 \\ .10 \end{pmatrix}$ and $\begin{pmatrix} .25 \\ .50 \\ .20 \end{pmatrix}$, respectively. The different costs of the company are described as the following linear combination:

$$x_1 \begin{pmatrix} .25 \\ .25 \\ .10 \end{pmatrix} + x_2 \begin{pmatrix} .25 \\ .50 \\ .20 \end{pmatrix} \in \mathbf{R}^3$$

b) We work with the augmented matrix corresponding to the vector equation below.

$$\left(\begin{array}{cc|c} .25 & .25 & 1000 \\ .25 & .50 & 1400 \\ .10 & .20 & 560 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 4000 \\ 1 & 2 & 5600 \\ 1 & 2 & 5600 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 4000 \\ 1 & 2 & 5600 \\ 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 4000 \\ 0 & 1 & 1600 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2400 \\ 0 & 1 & 1600 \\ 0 & 0 & 0 \end{array}\right)$$

The solution is $x_1 = 2400, x_2 = 1600$; that is, the budget will be used entirely by producing 2400 frisbees and 1600 boomerangs.

c) The vector equation is given by $x_1 \begin{pmatrix} .25 \\ .25 \\ .10 \end{pmatrix} + x_2 \begin{pmatrix} .25 \\ .50 \\ .20 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1400 \\ 560 \end{pmatrix}$

Problem 5.

Consider vectors $v_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

a) [4pts] Provide the definition of $\mathrm{Span}\{v_1, v_2, v_3\}$.

b) [2pts] Is vector $\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ in $\mathrm{Span}\{v_1, v_2, v_3\}$?

c) [4pts] (Unrelated) Write a system of linear equations whose solution set contains $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and is parallel to $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}$.

Solution.

a) The Span of v_1, v_2 and v_3 is a set containing all linear combinations of these vectors. In a formula,

$$Span\{v_1, v_2, v_3\} = \left\{x_1 v_1 + x_2 v_2 + x_3 v_3 \in \mathbf{R}^3 : x_1, x_2, x_3 \in \mathbf{R}\right\}$$

b) For any coefficients x_1, x_2, x_3, the second entry of $x_1 v_1 + x_2 v_2 + x_3 v_3$ is a zero; therefore, $\begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ is not in $\mathrm{Span}\{v_1, v_2, v_3\}$.

Alternatively, the augmented matrix associated to $x_1 v_1 + x_2 v_2 + x_3 v_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$ is

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 3 & 2 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -4/5 & -3 \\ 0 & 1 & 6/5 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right);$$

that is, it is equivalent to an incosistent reduced row echelon form.

c) Suppose we have a system $Ax = 0$ whose solution set is $Span\left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}$. Then the solutions satisfy

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

or equivalently,

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -3x_3 \\ x_3 &= x_3 \end{aligned}$$

Translating back to an augmented matrix we obtain

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

therefore the homogeneous system with $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ is a system with solution set $Span\left\{\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right\}$.

Don't plug in the translation by $p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ just yet. Instead, what we want to do is to define a system $Ax = b$ such that vector $p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is one of its solutions. We compute

$$Ap = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix},$$

and set $b = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$ Then, with the values above, the system $Ax = b$ translates to

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ x_2 + 3x_3 &= 4 \end{aligned}$$

and its solution set passes through $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and is parallel to $Span\left\{\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}\right\}$.