MATH 1553
MIDTERM EXAMINATION 2

Name		Section	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (calculators, notes, text, etc.) allowed.
- Please read carefully all statements and show all your work.
- You may cite any theorem or slide covered in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[2 points each]

Suppose that A is an $m \times n$ matrix. Let $T(x) = Ax$ be the linear transformation associated to A. Circle true for all statements that are always true, and circle false for statements that are ever false.

a) **T** **F** If A has more columns than rows then T is not one-to-one.

b) **T** **F** If the columns of A are linearly independent then T is onto.

c) **T** **F** The span of the columns of A equals the range of T.

d) **T** **F** The dimension of $\operatorname{Col} A$ is at most m.

e) **T** **F** The equation $\operatorname{rank} A + \dim \operatorname{Nul} A = n$ holds only when $m = n$.

Solution.

a) This is true. The RREF of A will necessarily have non-pivot columns.

b) False: let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution.

c) This is true. Both subsets contain exactly all linear combinations of the columns of A.

d) This is true. This true since columns of A are vectors in $\mathbf{R}^m$.

e) False: This is the Rank theorem, there is no restriction on m.

Problem 2.

Determine which of the following matrices have an inverse and explain why. When the inverse exists, compute A^{-1} and solve $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

a) $A = \begin{pmatrix} 0 & -2 \\ 1/2 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}$

c) **Aside:** Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, select one vector v_3 from:

$$\left\{ \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

that makes $\{v_1, v_2, v_3\}$ linearly independent. Justify your choice.

Solution.

a) Since $A^{-1} = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix}$, then $x = A^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Inverse Reason 1: Using the formula for the 2×2 case, if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $A^{-1} = det(A)^{-1}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$. In this case $det(A) = 0 \cdot 0 - (-1/2)(2) = 1$.

Inverse Reason 2: Using the algorithm

$$\left(\begin{array}{cc|cc} 0 & -2 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1/2 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = -2R_1} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & -2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 = .5R_2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 2 \\ 0 & 1 & -1/2 & 0 \end{array}\right)$$

Inverse Reason 3: If you were able to 'simply guess' A^{-1} then you must show your work verifying $AA^{-1} = I_n$.

b) This matrix has no inverse because its columns are multiple of each other.
Reason 2: In this case $det(A) = 3 \cdot (-1) - (-3)(1) = 0$, so the inverse does not exist.
Reason 3: Using the algorithm we cannot get the identity matrix on the left.

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

c) All systems $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} x = 0$, $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} x = 0$ and $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x = 0$ have exactly one solution, so each set of column vectors are independent.

On the other hand, the following are linear dependencies

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So either vector in $\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right\}$ makes the set $\{v_1, v_2, v_3\}$ linearly dependent.

Problem 3.

We have the following information about the linear transformations T and U:

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ x_2 \\ x_2 \end{pmatrix},$$

$$U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \text{ and } U\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}.$$

a) Give the standard matrix of T

b) Is T onto? Justify your answer.

c) Compute $U\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and justify your answer.

d) Which of the following is defined?

$$T \circ U(e_1) \qquad \text{or} \qquad U \circ T(e_1)$$

Solution.

a) Evaluate T on the unit vectors: $T(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $T(e_3) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then the standard matrix of T is $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

b) You can use any of the statements of the Onto Theorem to justify this. E.g. $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ can be reduced to a matrix with one row of zeros, then it does not have a pivot in each row and so T is not onto.

c) Since $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2e_1 - e_2$, then by the linearly of the transformation U we have

$$U\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2U(e_1) - U(e_2) = 2\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$$

d) The composition that makes sense is $T \circ U$. Note that the codomain of U coincides with the domain of T (both $\mathbf{R}^3$).

Problem 4.

Consider the following matrix and its reduced row echelon form:

$$A = \begin{pmatrix} -2 & 4 & 1 & -7 \\ 1 & -2 & 1 & -1 \\ 1 & -2 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for $\operatorname{Col} A$.

b) What is $\dim \operatorname{Nul} A$?

c) Is $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$ a basis for $\operatorname{Nul} A$? Justify your answer.

Solution.

a) A basis for $\operatorname{Col} A$ consists of the pivot colums of A:

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

b) The dimension of $\operatorname{Nul} A$ equals 2, this is the number of free variables in the reduced row echelon form. Or alternatively, we can use the Rank theorem.

c) Both $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ are contained in $\operatorname{Nul} A$. However, they are multiple of each other; thus they cannot be a basis.

Problem 5.

Which of the following are subspaces of $\mathbf{R}^3$? Justify why.

a) $V = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; ab = 0 \right\}$

b) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

c) The solution set of $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

d) The range of $T(x) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$.

e) Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix} \right\}$

Solution.

a) This is not a subspace of $\mathbf{R}^3$: it is not closed under addition. For instance,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ are in } V\text{, but } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ is not.}$$

b) This is a subspace of $\mathbf{R}^3$: The subset satisfies all the conditions of a subspace.

c) This is not a subspace of $\mathbf{R}^3$ because it is not the solution set to the homogeneous equation $Ax = 0$.

d) This is a subspace of $\mathbf{R}^3$: the codomain of T is $\mathbf{R}^3$ and the range is always a subspace.

e) This is a subspace of $\mathbf{R}^3$: the span of any collection of vectors in $\mathbf{R}^3$ is always a subspace of $\mathbf{R}^3$.

Scoring Table

Please do not write on this area.

1	2	3	4	5	Total

[Scratch work]