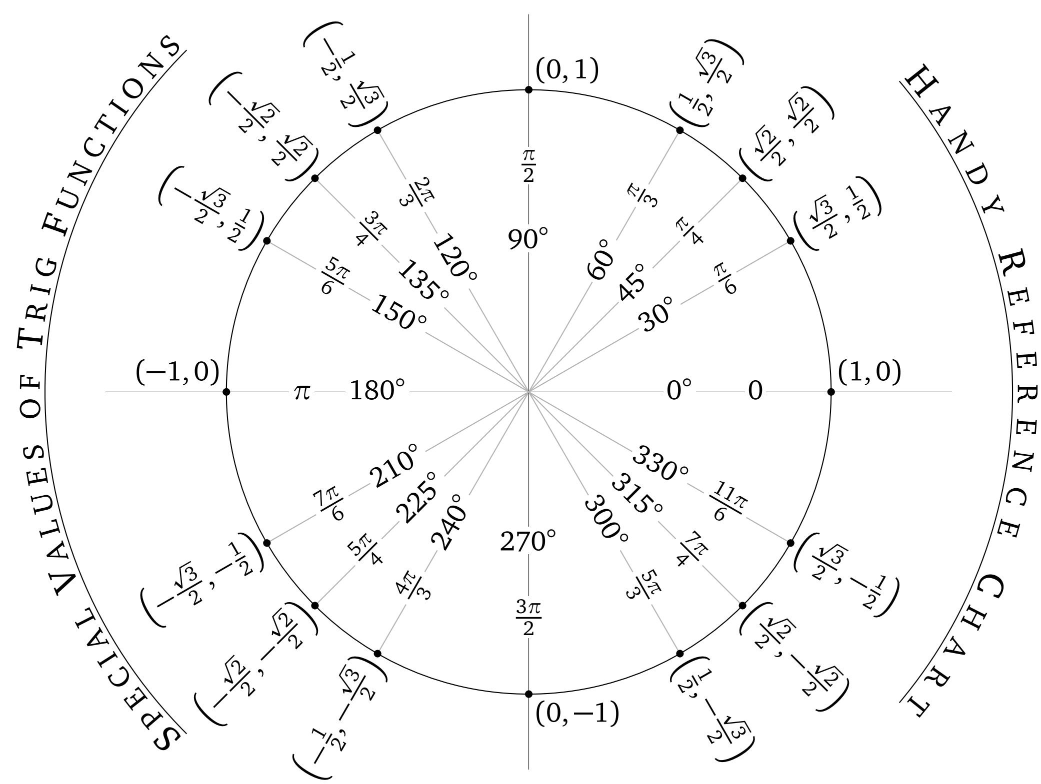
MATH 1553
MIDTERM EXAMINATION 3

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text,calculator,...) allowed.
- You may use the chart below to find unit vector coordinates.
- Please show your work unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



Problem 1.

[2 points each]

In this problem, if the statement is always true, circle **T**; otherwise, circle **F**. Let A and B be $n \times n$ matrices with real entries.

a) **T** **F** For $c \neq 0$, $\det(cA) = c\det(A)$.

b) **T** **F** If v is a complex eigenvector of A then so is $\overline{v}$.

c) **T** **F** If B has real eigenvalues whose geometric multiplicity adds to n, then B is diagonalizable.

d) **T** **F** If A is not diagonalizable then A has a non-real eigenvalue.

e) **T** **F** The zero vector is an eigenvector of A with eigenvalue 0.

Solution.

a) False: The constant factors out with an n-th power: $\det(cA) = c^n \det(A)$.

b) True. Complex eigenvectors comes in conjugate pairs.

c) True.

d) False: For example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable and has no non-real eigenvalue.

e) False. Eigenvectors by definition must be non-zero vectors.

Problem 2.

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

a) [4 points] Find the real eigenvalues of A

b) [3 points] For each real eigenvalue of A, find a basis for the corresponding eigenspace.

c) [3 points] Is A diagonalizable? Why or why not?

Solution.

a) We compute the characteristic polynomial by expanding along the last row:

$$\begin{aligned} f(\lambda) &= \det \begin{pmatrix} 1-\lambda & -1 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)\det \begin{pmatrix} 1-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)^2\big((1-\lambda)^2+1\big) \\ &= (2-\lambda)(\lambda^2-2\lambda+2) \end{aligned}$$

The roots are 2 (with multiplicity 2) and $\lambda = 1 \pm i$ (with multiplicity 1). So this matrix has only one real eigenvalue: 2.

b) We compute the 2-eigenspace by solving $(A-2I)x = 0$:

$$A - 2I = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The parametric vector form of the general solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, so a basis for the 2-eigenspace is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Problem 3.

In this problem, briefly explain your answers.

a) [2 points] If A and B are similar and $\det(B) = 3$, compute $\det(A)$

b) [3 points] Suppose v and w are eigenvectors of A both with eigenvalue 2. Is $3v + 2w$ an eigenvector of A? Why or why not?

c) [3 points] Find a 2×2 rotation-scaling matrix C that rotates by $45°$ and scales by $\sqrt{2}$.

d) [2 points] Suppose A is 4×4 matrix with a 2-eigenspace of dimension 3 and $\mathrm{Nul}(B)$ of dimension 1. Find a diagonal matrix that is similar to A.

Solution.

a) This means there is invertible P such that $A = PBP^{-1}$, then

$$\det(A) = \det(P)\det(B)\det(P^{-1}) = (\det(P)\det(P^{-1}))\det(B) = \det(B).$$

b) We can simply compute $A(3v + 2w) = 3Av + 2Aw = 6v + 4w = 2(3v + 2w)$. Thus, $3v + 2w$ is an eigenvector with eigenvalue 2.

c) Using the chart on the front of the exam: $Ae_1 = \begin{pmatrix} \cos(\pi/4) \\ \sin(\pi/4) \end{pmatrix}$ and $Ae_2 = \begin{pmatrix} \cos(3\pi/4) \\ \sin(3\pi/4) \end{pmatrix}$.

Therefore, the matrix is $\begin{pmatrix} \cos(3\pi/4) & \cos(3\pi/4) \\ \sin(\pi/4) & \sin(3\pi/4) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Alternatively, you can reference a theorem or slide shown in class.

d) First, since $\mathrm{Nul}(A)$ has dimension 1, then the 0-eigenspace of B has dimension 1. Thus, the geometric multiplicities of 0 and 2 add up to 4; therefore, if a diagonal matrix is similar to A, then there are 3 diagonal entries equal to 2 and the rest are all zeros.

Eg. A is similar to $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

Problem 4.

Let $A = \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix}$

a) [5 points] Diagonalize the matrix $A = PDP^{-1}$ (Obtain matrices P and D)

b) [2 points] Compute $A^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) [2 points] Give a formula for $A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d) [1 points] Give a formula for D^n.

Solution.

a) First (2 points): Computing the characteristic polynomial we have

$$\det \begin{pmatrix} .8-\lambda & .4 \\ .2 & .6-\lambda \end{pmatrix} = (\lambda - .8)(\lambda - .6) - .08 = \lambda^2 - 1.4\lambda + .48 - 0.8$$

Using the quadratic formula to find the eigenvalues we obtain

$$\lambda = \frac{1.4 \pm \sqrt{1.4^2 - 4(.4)}}{2} = \frac{1.4 \pm .6}{2} = \begin{cases} 1 \\ .4 \end{cases}$$

Second (2 points): Find eigenvectors for both eigenvalues. For $\lambda = 1$:

$$A - I = \begin{pmatrix} -.2 & .4 \\ \star & \star \end{pmatrix} \implies \begin{pmatrix} .4 \\ .2 \end{pmatrix} \text{ is an eigenvector}$$

For $\lambda = .4$:

$$A - .4I = \begin{pmatrix} .4 & .4 \\ \star & \star \end{pmatrix} \implies \begin{pmatrix} .4 \\ -.4 \end{pmatrix} \text{ is an eigenvector}$$

Third (1 point): Write down the decomposition $A = PDP^{-1}$ where

$$P = \begin{pmatrix} .4 & .4 \\ .2 & -.4 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & .4 \end{pmatrix}$$

b) We can see that $.4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} .4 \\ -.4 \end{pmatrix}$ is a multiple of the eigenvector we found for the eigenvalue .4. So that $A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = .4 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Applying this equality three times we get $A^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (.4)^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

c) In general, for $A^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (.4)^n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

d) Since $D = \begin{pmatrix} 1 & 0 \\ 0 & .4 \end{pmatrix}$, we have that $D^n = \begin{pmatrix} 1 & 0 \\ 0 & .4^n \end{pmatrix}$.

Problem 5.

Consider the scaling-rotation matrix $C = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}$ and $A = \begin{pmatrix} \frac{-1+\sqrt{3}}{2} & \sqrt{3} \\ \frac{\sqrt{3}}{2} & \frac{-1-\sqrt{3}}{2} \end{pmatrix}$.
For the following problems, you can use that

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{-1}{2} & 1 \end{pmatrix}$$

a) [1 point] Are A and C similar? Why or why not?

b) [3 points] Find a complex eigenvalue of A with its eigenvector.

c) [2 points] Write down the formula for the matrix of a rotation by angle θ composed with scaling by c. (No justification needed).

d) [2 points] By what factor does C scale?

e) [2 points] By what angle does C rotate?

Solution.

a) Yes. Because $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{-1}{2} & 1 \end{pmatrix}$ are inverse of each other.

b) Using the formula from class we have that $A = PCP^{-1}$ where

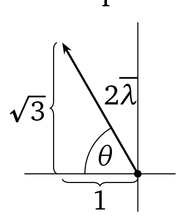
$$P = \begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix} \qquad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

For some eigenvector v with eigenvalue λ. So one eigenvector is $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$ with eigenvalue $\frac{-1-\sqrt{3}i}{2}$.

c) The matrix is $c \begin{pmatrix} \cos(\theta) & -sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

d) The matrix C has already the form above with $\theta = 2\pi/3$ and $c = 1$, so scaling is by 1.

e) Alternatively, if we use the decomposition of A, we need to find the argument of $\overline{\lambda} = \frac{-1}{2} + \frac{\sqrt{3}i}{2}$. We draw a picture:



$$\theta = \frac{\pi}{3} \text{ (trig identity)}$$
$$\text{argument of } \overline{\lambda} = \pi - \theta = \frac{2\pi}{3}$$

The matrix C rotates by $2\pi/3$.

Scoring Table

Please do not write on this area.

1	2	3	4	5	Total

[Scratch work]