Depth of high-degree vertices in Random recursive trees

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joint work with Louigi Addario-Berry

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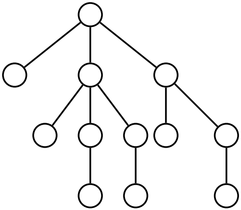
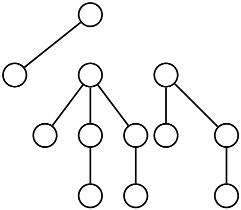
AMS Fall Eastern Sectional Meeting 2016

Underlying idea





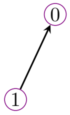
Random recursive trees/
Kingman's coalescent (Union-Find tree) **duality**.

Recursive trees: Standard construction

T_n is a rooted labelled tree on $V(T_n) = \{0, \ldots, n-1\}$.

Given T_n, construct T_{n+1}:

- Add a vertex labelled n,
- Select $j \in V(T_n)$ uniformly at random,
- Add the edge $n \to j$.



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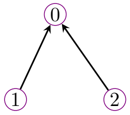

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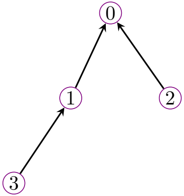

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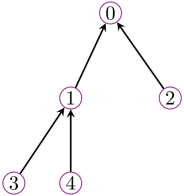

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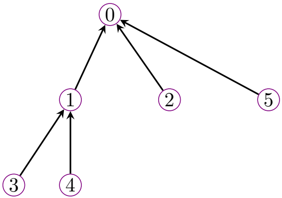

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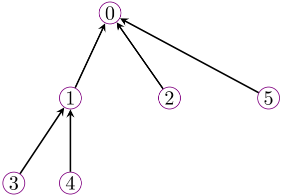
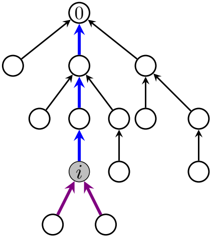


Facts. - Vertex labels are increasing along root-to-leaf paths.
- There are $(n-1)!$ possible outcomes for T_n.

Degree and depth

For $i \in V(T_n)$, let

$$\deg_n(i) = \#\{j > i : j \to i \text{ in } T_n\},$$
$$\mathrm{ht}_n(i) = \mathrm{dist}(0, i)$$



Depth of final vertex

Theorem. (Devroye, 1988, Mahmoud 1991) As $n \to \infty$,

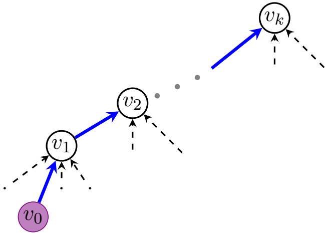
$$\frac{\mathrm{ht}_n(n-1) - \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

Proof's idea: Theory of records

$$v_0 = n-1, v_{i+1} = \lfloor v_i U_i \rfloor,$$

$$\mathrm{ht}_n(n-1) = \min\{i : v_i = 0\}$$

$$\stackrel{\mathcal{L}}{=} \sum_{k=1}^{n-1} \mathrm{Ber}(1/k)$$



The Bernoulli variables are independent.

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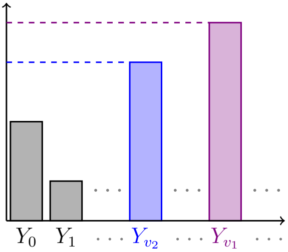

Y_v i.i.d. $\mathrm{Unif}(0,1)$

Depth of a uniformly chosen vertex

Theorem. (Devroye, 1988, Mahmoud 1991) As $n \to \infty$,

$$\frac{\mathrm{ht}_n(n-1) - \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

Corollary. Choose **uniformly** a vertex $\mathbf{u} \in T_n$. As $n \to \infty$,

$$\frac{\mathrm{ht}_n(\mathbf{u}) - \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{L}} \quad N(0,1)$$

What can we say about high-degree vertices?

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Theorem (Devroye, Lu 1995) Let $\Delta_n = \max\{\deg_n(i) : i \in V(T_n)\}$.
As $n \to \infty$, a.s.

$$\frac{\Delta_n}{\log_2 n} \to 1.$$

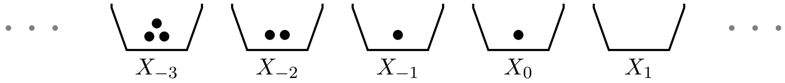
Asymptotic High-degree distribution

$$X_d^{(n)} = \#\{i \in [n] : \deg_n(i) = \lfloor \log_2 n \rfloor + d\}$$

Theorem (Addario-Berry,E. 2015$^+$) Let $n = 2^k$. Jointly for all $d \in \mathbb{Z}$, as $k \to \infty$

$$X_d^{(n)} \xrightarrow{\mathcal{L}} W_d;$$

where the W_d are independent Poisson r.v.'s with mean $2^{-(d+1)}$.

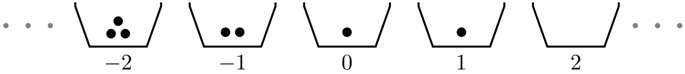


A marked point process on $\mathbb{Z} \times \mathbb{R}$

Let $2\alpha = \log_2 e$, then $2\alpha \ln n = \log_2 n$.

$$\bullet = \left(\deg_n(v) - \lfloor \log_2 n \rfloor, \frac{\mathrm{ht}_n(v) - (1-\alpha)\ln n}{\sqrt{(1-\alpha-\alpha^2/2)\ln n}} \right)$$

$\mathcal{DH}^{(n)}$:



A marked point process on $\mathbb{Z} \times \mathbb{R}$

Theorem (E. 2015^+) Let $n = 2^k$. There is an explicit marked point process $\mathcal{MP}$, such that in the space of marked point processes on $\mathbb{Z} \times \mathbb{R}$, as $k \to \infty$

$$\mathcal{DH}^{(n)} \xrightarrow{\mathcal{L}} \mathcal{MP};$$

the marks are distributed as independent standard gaussian variables.

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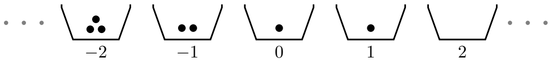

Conditional depth of a high-degree vertex

Proposition (E. 2015^+) Choose uniformly a vertex $\mathbf{u} \in V(T_n)$.
Fix $m \in \mathbb{Z}$, conditional on $\deg_n \mathbf{u} = \lfloor \log_2 n \rfloor + m$, as $n \to \infty$

$$\frac{\mathrm{ht}_n(\mathbf{u}) - (1-\alpha)\ln n}{\sqrt{(1-\alpha+\alpha^2/2)\ln n}} \xrightarrow{\mathcal{L}} N(0,1);$$

where $2\alpha = \log_2 e$.

Kingman's Coalescent

Fix $n \in \mathbb{N}$, for each $1 \leq t \leq n$ construct a forest of rooted labelled trees on $V(F_t) = \{1, \ldots, n\}$.

$$F_t = \{T_1^{(t)}, \ldots, T_{n-t+1}^{(t)}\}$$

Given F_t, construct F_{t+1}:

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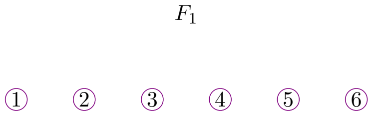
  directed to either tree with equal probability.

All choices are independent.



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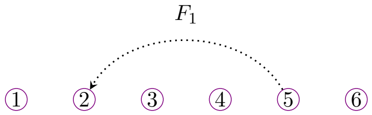

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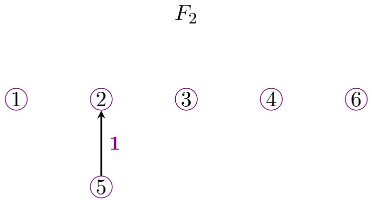

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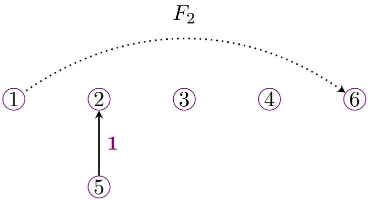

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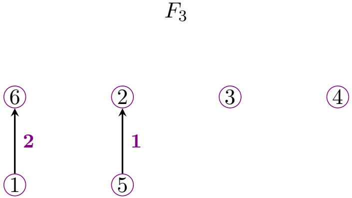

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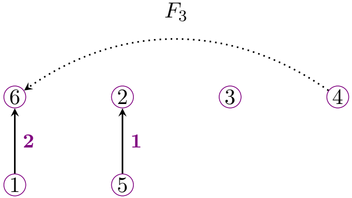

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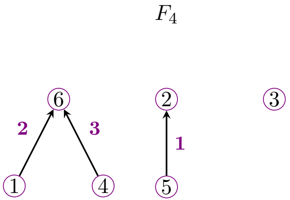

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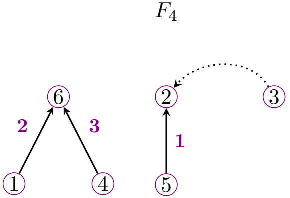

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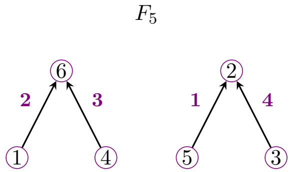

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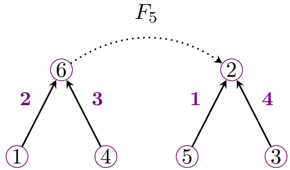

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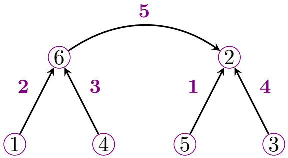
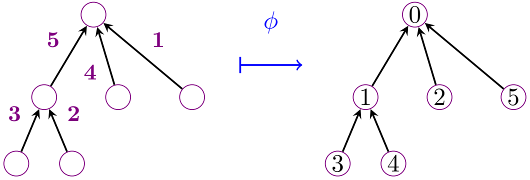


F_n

Recursive trees: Via Kingman's Coalescent

Lemma. There is a mapping ϕ such that $\phi(F_n) \stackrel{\mathcal{L}}{=} T_n$; furthermore, ϕ preserves the shape of F_n.



Proof's idea.

- Vertex labels are exchangeable.
- Edge labels are decreasing along root-to-leaf paths.
- There are $n!(n-1)!$ possible outcomes for F_n.

Selection set $\mathcal{S}^{(n)}$

$$\mathcal{S} = \mathcal{S}^{(n)} = \{t \leq n-1 : \text{Tree containing 1 merges at time } t\}$$

At step $t \in \mathcal{S}$, two trees are selected:

- One tree's root increases its degree and
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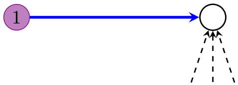

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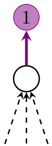

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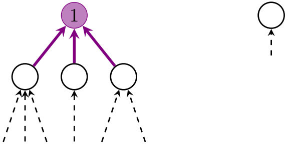

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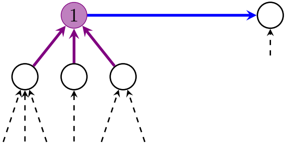

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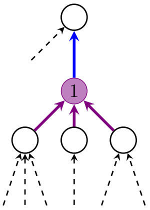

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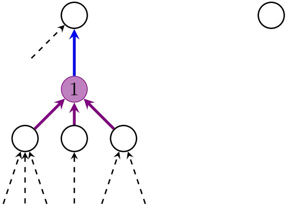

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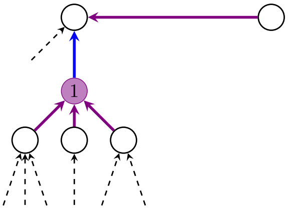

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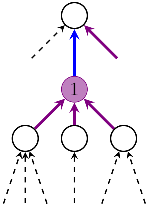

Selection set $\mathcal{S}^{(n)}$

$$\mathcal{S} = \mathcal{S}^{(n)} = \{t \leq n-1 : \text{Tree containing 1 merges at time } t\}$$

A favourable merge for 1 is when its tree's root increases its degree.

At step $t \in \mathcal{S}$, two trees are selected:

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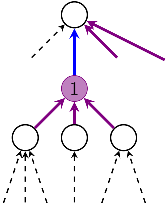

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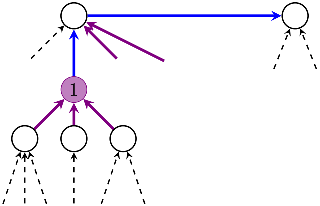

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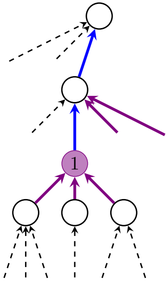


Degree and depth of vertex 1 in F_n

$$\mathcal{S} = \mathcal{S}^{(n)} = \{t \leq n-1 : \text{ Tree containing 1 merges at time } t\}$$

A favourable merge for 1 is when its tree's root increases its degree.

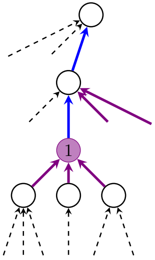
Proposition.

Depth = Total # unfavourable merges.

$$\mathrm{ht}_{F_n}(1) \stackrel{\mathcal{L}}{=} \mathrm{Bin}(|\mathcal{S}|, 1/2).$$

Degree = First streak favourable merges.

$$\deg_{F_n}(1) \stackrel{\mathcal{L}}{=} \min\{Geo(1/2), |\mathcal{S}|\}.$$



Asymptotic normality of $|\mathcal{S}^{(n)}|$ and $\mathrm{ht}_{F_n}(1)$

Theorem.

$$\frac{|\mathcal{S}^{(n)}| - 2\ln n}{\sqrt{2\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

$$\frac{\mathrm{ht}_{F_n}(1) - \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

Proof. Lindeberg conditítions for CLT are satisfied:

$$|\mathcal{S}^{(n)}| = \sum_{k=2}^{n} \mathrm{Ber}(2/k) \qquad \mathrm{ht}_{F_n}(1) = \sum_{k=2}^{n} \mathrm{Ber}(1/k).$$

Asymptotic normality of $\mathrm{ht}_{F_n}(1)$

A heuristic: Normal approximation of $|\mathcal{S}|$ and a Binomial r.v.

$$\mathrm{ht}_{F_n}(1) \stackrel{\mathcal{L}}{=} \mathrm{Bin}(|\mathcal{S}|, 1/2)$$

Asymptotic normality of $\mathrm{ht}_{F_n}(1)$

A heuristic: Normal approximation of $|\mathcal{S}|$ and a Binomial r.v.

$$\begin{aligned}\mathrm{ht}_{F_n}(1) &\overset{\mathcal{L}}{=} \mathrm{Bin}(|\mathcal{S}|, 1/2) \\ &\approx \frac{|\mathcal{S}|}{2} + \frac{B_{|\mathcal{S}|}}{2}\sqrt{|\mathcal{S}|}\end{aligned}$$

$$B_m \xrightarrow{\mathcal{L}} N_1 \overset{\mathcal{L}}{=} N(0,1)$$

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$$B_m \xrightarrow{\mathcal{L}} N_1 \overset{\mathcal{L}}{=} N(0,1)$$

$$S_n \xrightarrow{\mathcal{L}} N_2 \overset{\mathcal{L}}{=} N(0,1)$$

$$\frac{\mathbf{N_1} + \mathbf{N_2}}{\sqrt{\mathbf{2}}} \overset{\mathcal{L}}{=} N(0,1).$$

Conditional asymptotic normality of $\mathrm{ht}_{F_n}(1)$

Proposition. Fix $m \in \mathbb{Z}$, conditional on $\deg_{F_n}(1) = \lfloor \log_2 n \rfloor + m$,

$$\frac{\mathrm{ht}_{F_n}(1) - (1-\alpha)\ln n}{\sqrt{(1-\alpha+\alpha^2/2)\ln n}} \xrightarrow{\mathcal{L}} N(0,1)$$

where $2\alpha = \log_2 e$.

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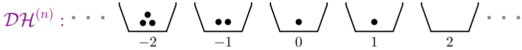
Proof's sketch: Given that $\deg_{F_n}(1) = \beta \ln n$, $\beta < 2$, then

$$|\mathcal{S}| - \deg_{F_n}(1) = (2-\beta)\ln n + S_n\sqrt{2\ln n} \qquad S_n \xrightarrow{\mathcal{L}} N(0,1).$$

Previous heuristic holds, now for $\mathrm{Bin}(|\mathcal{S}| - \deg_{F_n}(1), 1/2)$.

Extending the study to many vertices

$$\bullet = \left(\deg_n(v) - \lfloor \log_2 n \rfloor, \frac{\mathrm{ht}_n(v) - (1-\alpha)\ln n}{\sqrt{(1-\alpha-\alpha^2/2)\ln n}} \right)$$



Tools.

- Method of moments.
- Exchangeability of vertices in F_n,
- weak correlation between selection sets of distinct vertices.

Thanks!

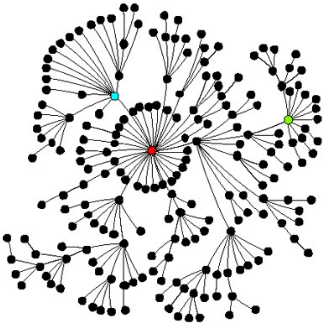


Imagen from scalefreenetworks, Flickr