Properties for extreme-valued degrees in recursive trees

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PhD Oral Defence

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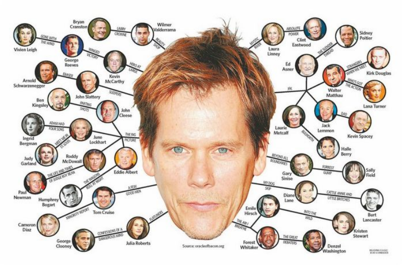
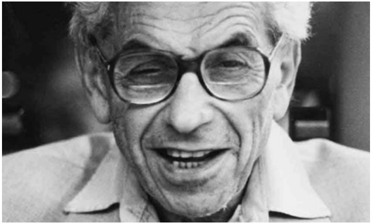
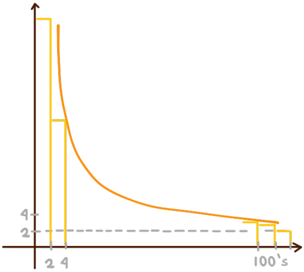
Motivation: Hubs in random networks







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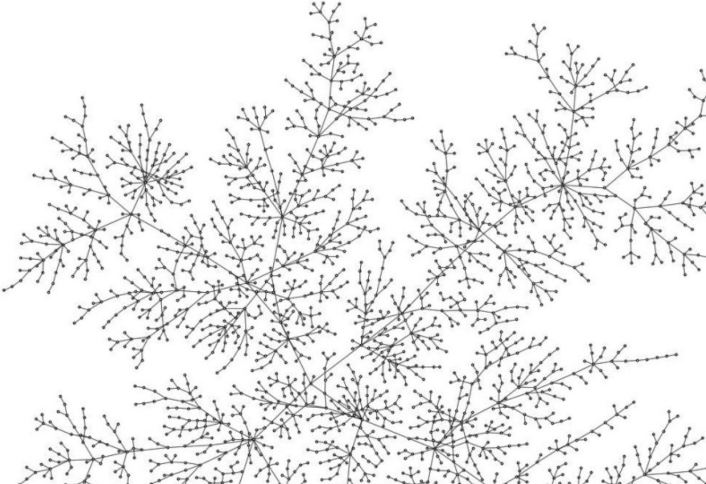
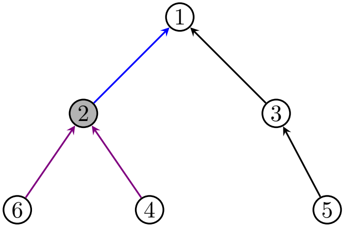
Tree growth processes

Image by Louigi A.-B.



Notation for trees: T

- Root / Leaves
- Children / Degree $\deg_T(\cdot)$
- Depth $\mathrm{ht}_T(\cdot)$ / Height
- Edges directed towards root.
- Vertices are labeled with $[n] = \{1, \ldots, n\}$.



Tree growth processes

Construct a tree growth process $(T_n,\ n \in \mathbb{N})$:

- T_1 is a single-vertex tree.
- For $n > 1$, build T_n from T_{n-1} adding: $\begin{cases} \text{vertex } n \\ \text{edge } n \to j \end{cases}$

$$\mathbb{P}(n \to j) = \frac{\beta \deg_{T_{n-1}}(j) + 1}{(\beta+1)(n-2)+1}$$

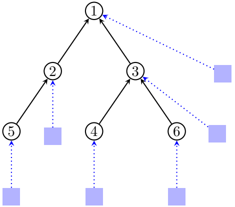
When $\beta > 0$: *The rich gets richer.*

Recursive Trees and Linear Pref. Attachment Trees

At any step new vertex n attaches to j

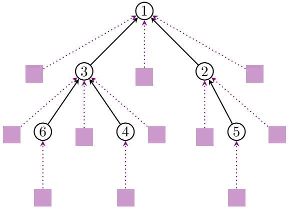
$$\mathbb{P}(n \to j) = \frac{1}{n-1}$$

New edge-connection uniform and independent of the past.



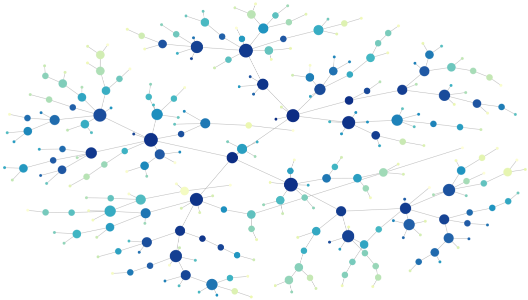
$$\mathbb{P}(n \to j) = \frac{\deg_{T_{n-1}}(j) + 1}{2n-1}$$

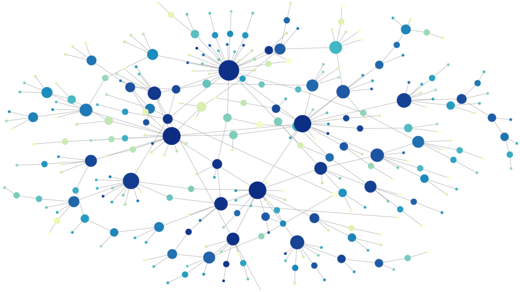
Visually: Construct a plane-oriented recursive tree, then forget about embedding.



Simulations

Recursive tree



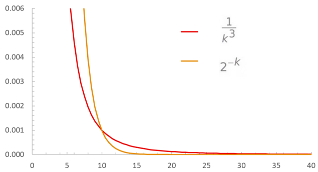


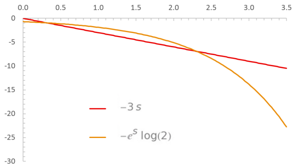
Linear Pref. Attachment tree

Geometric vs. Power law

- [Janson 05] Empirical probability: Given a tree T on n vertices. Select a uniformly random vertex u.

$$\mathbb{P}(\deg_T(u) = k-1) \approx \begin{cases} 2^{-k} & T \text{ recursive tree} \\ ck^{-3} & T \text{ linear pref. attachment tree} \end{cases}$$





Maximum degree

$$\Delta_n = \max\{\deg_{T_n}(i) : i \in [n]\}$$

[Devroye, Lu 1995] If T_n is a recursive tree. As $n \to \infty$, a.s.

$$\frac{\Delta_n}{\log n} \to 1.$$

[Mori 2005] If T_n is a linear pref. attachment tree, there is Δ r.v. such that, as $n \to \infty$, a.s.

$$\frac{\Delta_n}{\sqrt{n}} \to \Delta.$$

Maximum degree

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[Goh, Schmutz 2002] If T_n is a recursive tree, $n = 2^k$. For $i \in \mathbb{N}$ fixed

$$\mathbb{P}(\Delta_n - \log n < i) = \exp\{-2^{-i}\} + o(1).$$

Insertion depth

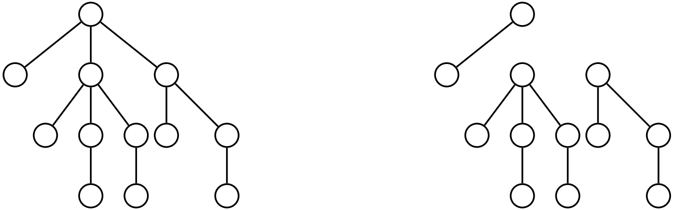
[Devroye, 1988, Mahmoud 1991] If T_n is a recursive tree. As $n \to \infty$,

$$\frac{\mathrm{ht}_{T_n}(n) - \ln n}{\sqrt{\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

[Mahmoud 1992] If T_n is a linear pref. attachment tree. As $n \to \infty$,

$$\frac{\mathrm{ht}_{T_n}(n) - (1/2)\ln n}{\sqrt{(1/2)\ln n}} \xrightarrow{\mathcal{L}} N(0,1).$$

Extreme-valued degrees in recursive trees

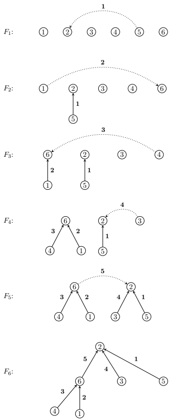


Kingman's Coalescent

Fix $n \in \mathbb{N}$, for each $1 \leq t \leq n$ construct a forest of rooted labelled trees on $V(F_t) = \{1, \ldots, n\}$. Given F_t, construct F_{t+1}:

- Uniformly choose two trees in F_t,
- Add an edge between the roots: directed to either tree with equal probability.

All choices are independent.



Lemma. There is a mapping ϕ such that

$$\phi(F_n) \stackrel{\mathcal{L}}{=} T_n;$$

furthermore, ϕ preserves the shape of F_n.

Degree and depth of vertex 1 in F_n

$$\mathcal{S} = \mathcal{S}^{(n)} = \{t \le n-1 : \text{Tree containing 1 merges at time } t\}$$

A favourable merge for 1 is when its tree's root increases its degree.

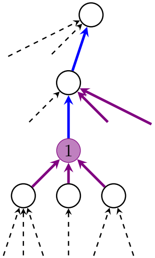
Proposition.

Depth = Total # unfavourable merges.

$$\mathrm{ht}_{F_n}(1) \stackrel{\mathcal{L}}{=} \mathrm{Bin}(|\mathcal{S}|, 1/2).$$

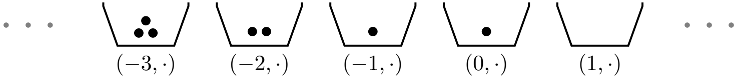
Degree = First streak favourable merges.

$$\deg_{F_n}(1) \stackrel{\mathcal{L}}{=} \min\{Geo(1/2), |\mathcal{S}|\}.$$



My research main focus: A marked point process

$$\bullet \;=\; \left(\deg(v) - \lfloor \log n \rfloor, \frac{\mathrm{ht}(v) - (1-\alpha)\ln n}{\sqrt{(1-\alpha/2)\ln n}} \right)$$



- ▷ Natural
  - \# vertices with same degree k have Poisson(2^{-k-1}) distribution
  - Depths have Gaussian fluctuations
- ▷ Good news
  - '# vertices'/depths become independent variables
- ▷ Surprising
  - Process of **vertex-overtakes** to become max-degree

Comparing to Linear Pref. Attachement

[Mori2005, Pekös, Röllin, Ross 2016] If T_n is a linear perf. attachment tree. As $n \to \infty$,

$$\left(\frac{\deg_{T_n}(i)}{\sqrt{n}}, \, i \geq 1 \right) \xrightarrow{\mathcal{L}} (B_i, i \geq 1) \, ,$$

where for all $k \geq 2$,

$$\sum_{i=1}^{k} B_i \stackrel{\mathcal{L}}{=} \left(\sum_{i=1}^{k} E_i \right)^{1/2}$$

$E_i \stackrel{\mathcal{L}}{=} \mathrm{Exp}(1)$ are iid., and

$$\frac{B_k}{\sum_{i=1}^{k} B_i} \stackrel{\mathcal{L}}{=} \mathrm{Beta}(2, 3\mathrm{k} - 4)$$

Summary of results

- **Poisson Point Process for near-maximum degree vertices**

  Number and their depth

- Conditional depth of high-degree vertices
- Tighten tails for maximum degree distribution ($\sim$Gumbel)

- **CLT's -rates of converge ($1 < c < \log e$)**

$$X_c = \{v \in [n], \deg_{T_n}(v) \geq c \ln n\}$$

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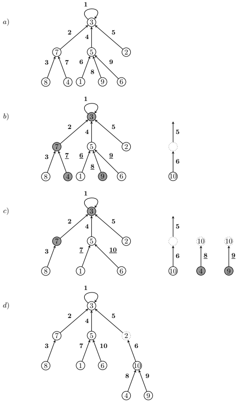
- High degrees of random recursive trees (joint with Louigi Addario-Berry). RSA 2017$^+$
- Depth of vertices with high degree in random recursive trees.
- Extremal values in recursive trees via a new tree growth process.

Summary

- ▷ Uniform vs. preferential attachment
  - Depth: Both logarithmic
  - Degrees: Geometric / Power laws
  - Max-deg vertex: Ever-changing / 'Fixed'
- ▷ Advantages of coalescent
  - Exchangeability
  - Decouple randomness
  - Depth/degree relation
- ▷ Further research
  - Depth of max-deg for branching processes
  - Novel dynamics for tree growth process



Thanks!

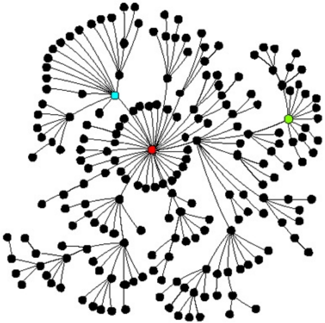


Image from scalefreenetworks, Flickr