Branching processes with cousin mergers and locality of hypercube's critical percolation

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joint work in progress with S. Penington, F. Skerman

Spectra, Algorithms and Random Walks on Random Networks
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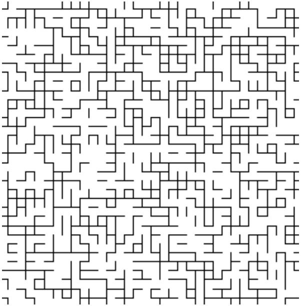
Making sense of formulas

Contents

Percolation

Given an *underlying graph*, keep each edge independently with prob. p

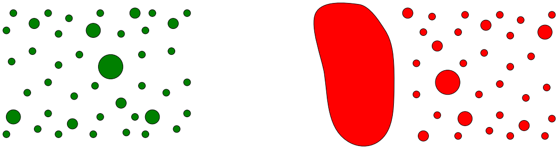


Critical Probability: *Edge density where an giant component appears*

- *Infinite graphs:* $p_c := \inf\{p : P(|C(0)| = \infty) > 0\}$

Phase transition for Erdős-Rényi $G_{n,p}$

Critical Probability: *Edge density where a giant component appears*



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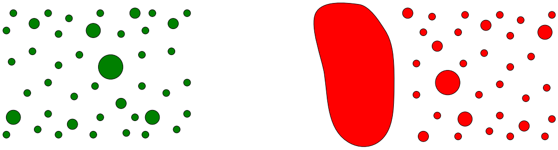

If $p = \frac{1}{n}(1+\epsilon)$, *whp largest component of size:*

- **Subcritical** $\epsilon^3 n \to -\infty$: $L_1(G_{n,p}) = O(\log n)$
- **Critical** $\epsilon^3 n \to a \in \mathbb{R}$: $L_1(G_{n,p}) = \Theta(n^{2/3})$
- **Supercritical** $\epsilon^3 n \to \infty$: $L_1(G_{n,p}) = \Theta(n)$

The critical window is of order $O(n^{-4/3})$

Percolation in finite graphs

First reference was Erdős-Rényi graphs $p_c = \frac{1}{n}$

- Finite graphs: $p_c :=$???
- How big can components be?

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A definition that *works*

For finite transitive graphs with V vertices and degree m, fix $\lambda \in (0,1)$.

Let $p_c = p_c(\lambda)$ solve

$$E_{p_c(\lambda)}[|C(0)|] = \lambda V^{1/3}$$

Fact:

$$p_c(\lambda_1) - p_c(\lambda_2) = O(m^{-1} V^{-1/3})$$

Why is $p_c = 1/n$?

Detour to Galton-Watson trees

- Indiv. v has random ξ_v children independently from rest.

$$Z_n = \#\text{indiv. at generation } n$$

Galton-Watson Survival

Average children $\mathbb{E}[\xi_v] = (1+\epsilon)$ determines

- $\epsilon \leq 0$: Extinction w.p. 1
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Markovian process: Each generation only depends on previous one.

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{v_i}$$

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Exploring components approximation:

- When graph has high dimension, exploration on giant component goes on forever.
- On $G_{n,p}$, each vertex sees $Bin(n-1, p)$ other vertices

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Heuristic for percolation: should use $p_c(n-1) \sim 1$ instead

If $\epsilon + O(n^{-1/3})$ then there is $\epsilon' = O(n^{-1/3})$ with

$$p = \frac{1}{n}(1+\epsilon) = \frac{1}{n-1}(1+\epsilon')$$

The case of the Hypercube Q^n

Borgs et al. ['05, '06]; Hosftad, Slade ['05, '06], Hofstad, Nachmias ['12,'14]

There exists rational coefficients a_k such that

$$p_c = \sum_{k=1}^{K} a_k n^{-k} + O(n^{-K-1})$$

In particular,

$$\begin{aligned} p_c &= \frac{1}{n} + \frac{1}{n^2} + \frac{7}{2n^3} + O(n^{-4}) \\ &= \frac{1}{n-1} + \frac{5}{2}(n-1)^{-3} + O(n^{-4}) \end{aligned}$$

- Based on lace expansion and triangle condition verification
- Window too small $O(n^{-1}2^{-1/3})$ to neglect any expansion term

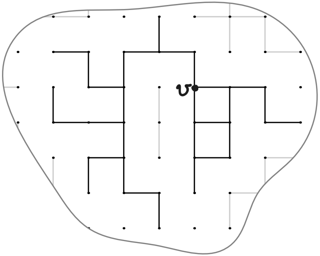
Exploration on lattice-like graphs

Goal: Count size of a vertex v component

Exploration tracks:

- Explored vertices D_k
- To-explore vertices X_k

$$D_0 = \{v\}, \qquad X_0 = \emptyset$$



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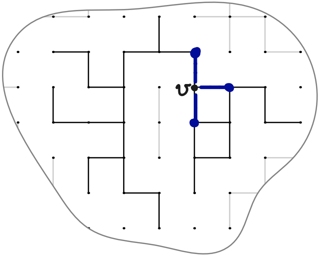

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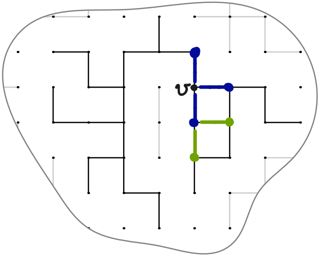

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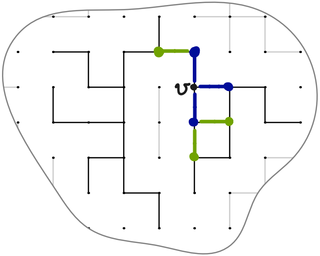

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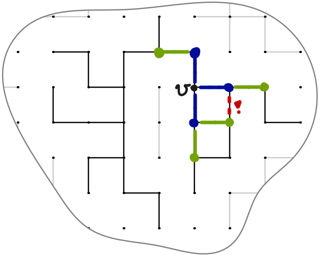
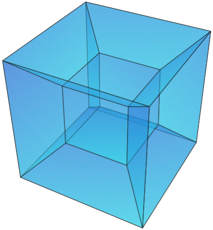


Branching process - collisions
$\approx$
Component size

The hypercube's local structure

$\{0,1\}^n$ Representation

- Sequences with n entries
- Crossing edge changes **one** entry:
  $(0, 0, 1, 0, \ldots, 0)$
- Smallest cycle has length 4



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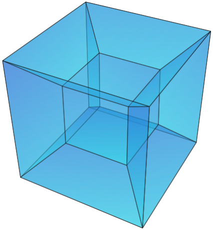

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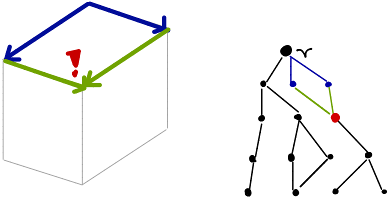
Simple collisions

- Parent: $(1, 0, 0, 0, \dots, 0)$
- Possible children:

  $(1, 0, 1, 0, \dots, 0)$

  $(1, 1, 0, 0, \dots, 0)$
- Possible grandkid:
  $(1, 1, 1, 0, \dots, 0)$



Two steps in exploration

Project: Heuristic to recover $c = \frac{5}{2}$

The local structure of hypercube predicts coefficients of critical p_c

- **'Guess'** $$p_c = (n-1)^{-1} + c(n-1)^{-3}$$
  and tune c via the survival threshold of a branching process

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- **'Guess'** $$p_c = (n-1)^{-1} + c(n-1)^{-3}$$ and tune c via the survival threshold of a branching process

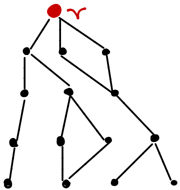
- **Design:** A *modified* Poisson branching process with suitable survival threshold.

Exploration	Hypercube dim. n	Branching
Average children	$1 + \frac{5}{2}(n-1)^{-2}$	$1 + \epsilon$
Cousin identification	$(n-1)^{-2}$	q

$$\mathrm{Bin}(n-1, p_c) \approx \mathrm{Poi}(1+\epsilon)$$
$$(n-1)p_c \approx 1 + \epsilon = 1 + cq$$

Branching process with cousin mergers

- Indiv. have $\xi_v \sim \mathrm{Poi}(1+\epsilon)$ children
- Independently with probability q, each pair of cousins becomes a single indiv.
- Multiple mergers allowed



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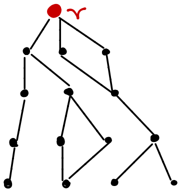

Difficulties:

Non-Markovian Process: $Z_0, Z_1, \ldots, Z_n$ not enough to obtain Z_{n+1}

Non-monotonicity: No straightforward coupling gives monotonicity of survival

Survival Gap

BP with cousin mergers (E., Penington, Skerman, '20$^+$)

If $\xi_v \sim \mathrm{Poi}(1+\epsilon)$, $\epsilon > 0$ suff. small, then merger prob. q determines

- $q \geq 2\epsilon + K\epsilon^2$: Extinction w.p. 1
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But ideal ratio was $q = \frac{2}{5}\epsilon$

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Partial Idea: Estimate average growth per generation

$$\mathbb{E}[Z_{n+1}] \approx \left(1 + \epsilon - \frac{q}{2}\right) \mathbb{E}[Z_n]$$

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Zoom-in on idea

Partial Idea: Estimate average growth per generation

$$\mathbb{E}[Z_{n+1}] \approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1+\epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)$$

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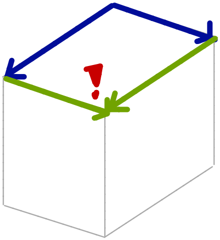

$\mathbb{E}[\#$ pairs of cousins per grandparent$]$

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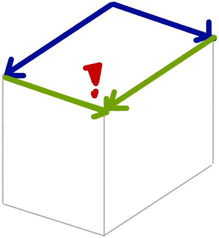

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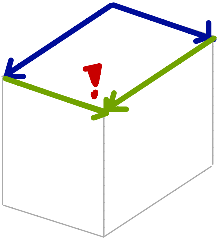

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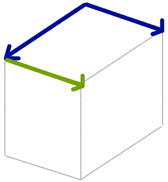
What went wrong?

Offspring distribution: Not all vertices can explore $n-1$ new edges

$$\mathbb{E}[Z_{n+1}] \approx (1+\epsilon)\mathbb{E}[Z_n] - \frac{q}{2}\mathbb{E}[Z_{n-1}] - q\mathbb{E}[Z_{n-2}] - q\mathbb{E}[Z_{n-3}]$$



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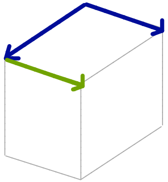

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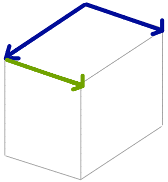

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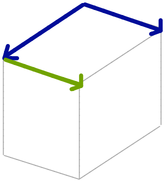

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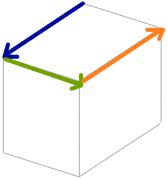

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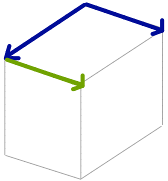

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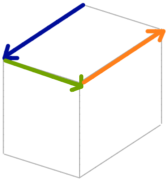

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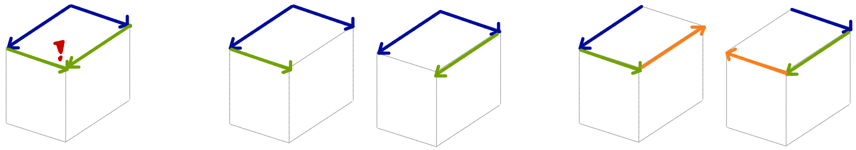
All these pairs give collisions with probability

$$(n-1)^{-2} \sim q$$

Refining the cousin mergers model



Process construction

From generation n to $n+1$:

1. Reproduction: Indiv. at generation n have children.
2. Deletions: Keep *authentic* children w.p. $(1-q)^{k_v}$
3. Collisions: Each pair of cousins flip biased coin,
4. Identification: of pairs of cousins.

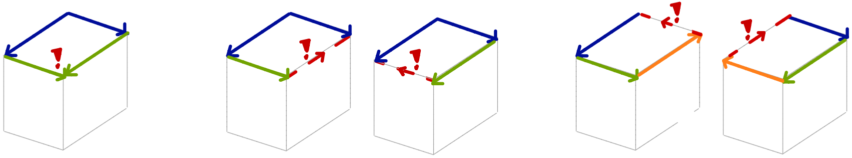
Survival gets the mysterious coefficients!

Refined BP with collisions (E., Penington, Skerman, '20$^+$)

If $\xi_v \sim \mathrm{Poi}(1+\epsilon)$, $\epsilon > 0$ suff. small, then collision prob. q determines

- $q \geq \frac{2}{5}\epsilon + K\epsilon^2$: Extinction w.p. 1
- $q \leq \frac{2}{5}\epsilon - K\epsilon^2$: Survival w. positive prob.

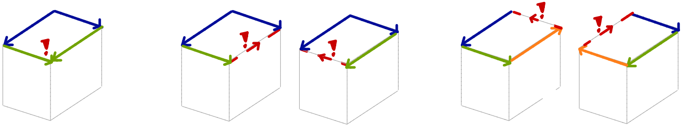
Partial Idea: There are collisions occurring 4 times as often



$$\mathbb{E}[Z_{n+1}] \approx \left(1 + \epsilon \quad -\frac{1}{2}q \quad -\frac{2}{2}q \quad -\frac{2}{2}q\right)(1 + O(\epsilon))\mathbb{E}[Z_n].$$

Summary

- We obtain a survival threshold for a variant of a branching process that mimics hypercube's exploration near criticality



- This sheds light on structures determining its critical probability

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