Branching processes with cousin mergers and locality of hypercube’s critical percolation

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Making sense of formulas
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   ▶ Critical probability
   ▶ Exploration of components

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   ▶ Critical probability expansions
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Percolation

Given an underlying graph, keep each edge independently with prob. $p$

Critical Probability: Edge density where an giant component appears

- Infinite graphs: $p_c := \inf \{ p : P(|C(0)| = \infty) > 0 \}$
Phase transition for Erdős-Rényi $G_{n,p}$

**Critical Probability:** *Edge density where a giant component appears*

![Diagram showing phase transition](image)

- **Subcritical** \( \epsilon \to -\infty \): \( L_1(G_{n,p}) = O(\log n) \)
- **Critical** \( \epsilon \to a \in \mathbb{R} \): \( L_1(G_{n,p}) = \Theta(n^{2/3}) \)
- **Supercritical** \( \epsilon \to \infty \): \( L_1(G_{n,p}) = \Theta(n) \)
Phase transition for Erdős-Rényi $G_{n,p}$

**Critical Probability:** *Edge density where a giant component appears*

If $p = \frac{1}{n}(1 + \epsilon)$, whp largest component of size:

- **Subcritical** $\epsilon^3 n \to -\infty$: $L_1(G_{n,p}) = O(\log n)$
- **Critical** $\epsilon^3 n \to a \in \mathbb{R}$: $L_1(G_{n,p}) = \Theta(n^{2/3})$
- **Supercritical** $\epsilon^3 n \to \infty$: $L_1(G_{n,p}) = \Theta(n)$

The critical window is of order $O(n^{-4/3})$
Percolation in finite graphs

First reference was Erdős-Rényi graphs \( p_c = \frac{1}{n} \)

- Finite graphs: \( p_c := ??? \)
- How big can components be?
Percolation in finite graphs

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- Finite graphs: $p_c := ???$
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**A definition that works**
For finite transitive graphs with $V$ vertices and degree $m$, fix $\lambda \in (0, 1)$.

Let $p_c = p_c(\lambda)$ solve

$$E_{p_c(\lambda)}[|C(0)|] = \lambda V^{1/3}$$

Fact:

$$p_c(\lambda_1) - p_c(\lambda_2) = O(m^{-1}V^{-1/3})$$
Why is $p_c = 1/n$?

Detour to Galton-Watson trees

- Indiv. $v$ has random $\xi_v$ children independently from rest.

\[ Z_n = \#\text{indiv. at generation } n \]

Galton-Watson Survival

Average children $\mathbb{E}[\xi_v] = (1 + \epsilon)$ determines

- $\epsilon \leq 0$: Extinction w.p. 1
- $\epsilon > 0$: Survival with positive prob.
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Markovian process: Each generation only depends on previous one.

\[ Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{v_i} \]
Why is $p_c = 1/n$?

Exploring components approximation:

- When graph has high dimension, exploration on giant component goes on forever.
- On $G_{n,p}$, each vertex sees $Bin(n - 1, p)$ other vertices
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Exploring components approximation:
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Heuristic for percolation: should use $p_c(n - 1) \sim 1$ instead

If $\epsilon + O(n^{-1/3})$ then there is $\epsilon' = O(n^{-1/3})$ with

$$p = \frac{1}{n}(1 + \epsilon) = \frac{1}{n-1}(1 + \epsilon')$$
The case of the Hypercube $Q^n$

Borgs et al. ['05, '06]; Hosftad, Slade ['05, '06], Hofstad, Nachmias ['12,'14]

There exists rational coefficients $a_k$ such that

$$p_c = \sum_{k=1}^{K} a_k n^{-k} + O(n^{-K-1})$$

In particular,

$$p_c = \frac{1}{n} + \frac{1}{n^2} + \frac{7}{2n^3} + O(n^{-4})$$

$$= \frac{1}{n - 1} + \frac{5}{2}(n - 1)^{-3} + O(n^{-4})$$

- Based on lace expansion and triangle condition verification
- Window too small $O(n^{-1}2^{-1/3})$ to neglect any expansion term
Exploration on lattice-like graphs

**Goal:** Count size of a vertex $v$ component

**Exploration tracks:**
- Explored vertices $D_k$
- To-explore vertices $X_k$

$$D_0 = \{v\}, \quad X_0 = \emptyset$$
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At each step $k$:
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2. Add **not explored** neighbors of $w$ to $X_k$
3. Move $w \in D_k$
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Branching process - collisions

\( \approx \)

Component size
The hypercube’s local structure

\[ \{0, 1\}^n \] Representation

- Sequences with \( n \) entries
- Crossing edge changes \textbf{one} entry:
  \( (0, 0, 1, 0, \ldots, 0) \)
- Smallest cycle has length 4

Two steps in exploration
The hypercube’s local structure

\{0, 1\}^n \textbf{Representation}

- Sequences with \( n \) entries
- Crossing edge changes \textbf{one} entry:
  \((0, 0, 1, 0, \ldots, 0)\)
- Smallest cycle has length 4

\textbf{Simple collisions}

- Parent: \((1, 0, 0, 0, \ldots, 0)\)
- Possible children:
  \((1, 0, 1, 0, \ldots, 0)\)
  \((1, 1, 0, 0, \ldots, 0)\)
- Possible grandkid:
  \((1, 1, 1, 0, \ldots, 0)\)

Two steps in exploration
Project: Heuristic to recover \( c = \frac{5}{2} \)

The local structure of hypercube predicts coefficients of critical \( p_c \)

- ‘Guess’  
  \[ p_c = (n - 1)^{-1} + c(n - 1)^{-3} \]
  and tune \( c \) via the survival threshold of a branching process
Project: Heuristic to recover $c = \frac{5}{2}$

"The local structure of hypercube predicts coefficients of critical $p_c$"

- ‘Guess’
  
  $$p_c = (n - 1)^{-1} + c(n - 1)^{-3}$$
  
  and tune $c$ via the survival threshold of a branching process

- **Design:** A *modified* Poisson branching process with suitable survival threshold.

<table>
<thead>
<tr>
<th>Exploration</th>
<th>Hypercube dim. $n$</th>
<th>Branching</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average children</td>
<td>$1 + \frac{5}{2}(n - 1)^{-2}$</td>
<td>$1 + \epsilon$</td>
</tr>
<tr>
<td>Cousin identification</td>
<td>$(n - 1)^{-2}$</td>
<td>$q$</td>
</tr>
</tbody>
</table>

$$\text{Bin}(n - 1, p_c) \approx \text{Poi}(1 + \epsilon)$$

$$(n - 1)p_c \approx 1 + \epsilon = 1 + cq$$
Branching process with cousin mergers

- Indiv. have $\xi_v \sim \text{Poi}(1 + \epsilon)$ children
- Independently with probability $q$, each pair of cousins becomes a single indiv.
- Multiple mergers allowed
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**Difficulties:**

**Non-Markovian Process:** $Z_0, Z_1, \ldots, Z_n$ not enough to obtain $Z_{n+1}$

**Non-monotonicity:** No straightforward coupling gives monotonicity of survival
Survival Gap

**BP with cousin mergers (E., Penington, Skerman, '20+)**

If $\xi_v \sim \text{Poi}(1 + \epsilon)$, $\epsilon > 0$ suff. small, then merger prob. $q$ determines:

- $q \geq 2\epsilon + K\epsilon^2$: Extinction w.p. 1
- $q \leq 2\epsilon - K\epsilon^2$: Survival w. positive prob.

**Partial Idea:** Estimate average growth per generation

$$E[Z_{n+1}] \approx (1 + \epsilon - q^2) E[Z_n] \approx (1 + \epsilon) E[Z_n] - q^2 E[Z_{n-1}]$$

But ideal ratio was $q = \frac{2}{5}\epsilon$
Survival Gap

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$$\mathbb{E}[Z_{n+1}] \approx \left(1 + \epsilon - \frac{q}{2}\right) \mathbb{E}[Z_n]$$
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But ideal ratio was $q = \frac{2}{5}\epsilon$

Partial Idea: Estimate average growth per generation

$$
\mathbb{E}[Z_{n+1}] \approx \left(1 + \epsilon - \frac{q}{2}\right) \mathbb{E}[Z_n] \\
\approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2} \mathbb{E}[Z_{n-1}]
$$
Partial Idea: Estimate average growth per generation

$$
\mathbb{E}[Z_{n+1}] \approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1 + \epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)
$$
Partial Idea: Estimate average growth per generation

$$\mathbb{E}[Z_{n+1}] \approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1 + \epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)$$

Recall. If $X \sim \text{Poi}(\lambda)$, then $\mathbb{E}[X(X - 1)] = \lambda^2$
Zoom-in on idea

**Partial Idea:** Estimate average growth per generation

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\mathbb{E}[Z_{n+1}] \approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1 + \epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)
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**Recall.** If \( X \sim \text{Poi}(\lambda) \), then \( \mathbb{E}[X(X - 1)] = \lambda^2 \)

\( \mathbb{E}[\# \text{ pairs of cousins per grandparent}] \)
**Partial Idea:** Estimate average growth per generation

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\mathbb{E}[Z_{n+1}] \approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2}(1 + \epsilon)^4\mathbb{E}[Z_{n-1}] + O(\epsilon^2)
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**Recall.** If \(X \sim \text{Poi}(\lambda)\), then \(\mathbb{E}[X(X - 1)] = \lambda^2\)

\[
\mathbb{E}[\# \text{ pairs of cousins per grandparent}] = \mathbb{E}[\# \text{ pairs of children } \{v_1, v_2\}] \\
\quad \cdot \mathbb{E}[\xi_{v_1}]\mathbb{E}[\xi_{v_2}]
\]
**Zoom-in on idea**

**Partial Idea:** Estimate average growth per generation

\[
E[Z_{n+1}] \approx (1 + \epsilon)E[Z_n] - \frac{q}{2} \left(1 + \epsilon\right)^4 E[Z_{n-1}] + O(\epsilon^2)
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**Recall.** If \(X \sim \text{Poi}(\lambda)\), then \(E[X(X - 1)] = \lambda^2\)

\[
E[\# \text{ pairs of cousins per grandparent}] = E[\# \text{ pairs of children } \{v_1, v_2\}] \\
\cdot E[\xi_{v_1}]E[\xi_{v_2}] \\
= \frac{(1 + \epsilon)^2}{2} \cdot (1 + \epsilon)(1 + \epsilon)
\]
What went wrong?

**Offspring distribution:** Not all vertices can explore $n - 1$ new edges

\[
\mathbb{E}[Z_{n+1}] \approx (1 + \epsilon)\mathbb{E}[Z_n] - \frac{q}{2}\mathbb{E}[Z_{n-1}] - q\mathbb{E}[Z_{n-2}] - q\mathbb{E}[Z_{n-3}]
\]

\[
\mathbb{E}[\# \text{ pairs of aunt-niece per grandparent}]
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\mathbb{E}[\# \text{ pairs of aunt-niece per grandparent}] = \mathbb{E}[\# \text{ pairs of children } (v_1, v_2)] \\
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and

\[
\mathbb{E}[\# \text{ greatgrandchildren per indiv.}] = (1 + \epsilon)^4
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and

\[
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\]

All these pairs give collisions with probability

\[
(n - 1)^{-2} \sim q
\]
Refining the cousin mergers model

Process construction

From generation $n$ to $n + 1$:

1. **Reproduction**: Indiv. at generation $n$ have children.
2. **Deletions**: Keep *authentic* children w.p. $(1 - q)^{k_v}$
3. **Collisions**: Each pair of cousins flip biased coin,
4. **Identification**: of pairs of cousins.
Survival gets the mysterious coefficients!

Refined BP with collisions (E., Penington, Skerman, '20+)

If $\xi_v \sim \text{Poi}(1 + \epsilon)$, $\epsilon > 0$ suff. small, then collision prob. $q$ determines

- $q \geq \frac{2}{5} \epsilon + K\epsilon^2$: Extinction w.p. 1
- $q \leq \frac{2}{5} \epsilon - K\epsilon^2$: Survival w. positive prob.

Partial Idea: There are collisions occurring 4 times as often

$$
\mathbb{E}[Z_{n+1}] \approx \begin{pmatrix}
1 + \epsilon & -\frac{1}{2}q & -\frac{2}{2}q & -\frac{2}{2}q \\
\end{pmatrix} (1 + O(\epsilon))\mathbb{E}[Z_n].
$$
Summary

- We obtain a survival threshold for a variant of a branching process that mimics hypercube’s exploration near criticality.

- This sheds light on structures determining its critical probability.

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