A branching process with deletions and mergers that matches the threshold for hypercube percolation

Laura Eslava, Sarah Penington and Fiona Skerman

Percolation Today
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Hypercube percolation: a brief story

- Connectivity threshold

  Burtin '77, Erdős, Spencer '79, Bollobás '84
- Emergence of giant component (transition at $p = 1/d$)

  Atjai, Komlós, Szemerédi '82
- Emergence of giant component (giant for $p \geq 1/(d-1)$)

  Bollobás, Kohayakawa, Łuczak '92
- Formal introduction of p_c Borgs et al. '05, '06
- Asymptotic expansion of p_c Hosftad, Slade '05, '06
- Unlacing the hypercube Hosftad, Nachmias '14

Previously, on $\mathbb{Z}^d$:

- First three terms of $p_c(\mathbb{Z}^d)$ expansion Hara, Slade '93

Expansion of critical probability

Hosftad, Slade ['05, '06]

Let $\mathbb{G}$ denote either Q_d or $\mathbb{Z}^d$ and let $\Omega = d$ or $\Omega = 2d$, respectively.

For all $K > 0$, there exists rational coefficients a_k such that

$$p_c(\mathbb{G}) = \sum_{k=1}^{K} a_k \Omega^{-k} + O(\Omega^{-K-1})$$

and

$$\begin{aligned} p_c(\mathbb{G}) &= \Omega^{-1} + \Omega^{-2} + \frac{7}{2}\Omega^{-3} + O(\Omega^{-4}) \\ &= (\Omega - 1)^{-1} + \frac{5}{2}(\Omega - 1)^{-3} + O(\Omega^{-4}) \end{aligned}$$

It is believed that $p_c(Q_d)$ and $p_c(\mathbb{Z}^d)$ differ after third term.

A branching process with deletions and mergers

Let $p > -1, q \in [0,1]$. Let $\mathcal{G}(p,q) = (G_n, n \geq 0)$ with $G_n = (V_n, E_n)$

$$I_n = V_n \setminus V_{n-1} \text{ is the } n\text{th generation.}$$

For $v \in I_{n-1}$, let $k_v = \#\{w \in V_{n-1} : \ d_{G_n}(v,w) = 3\}$.

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Construction: $G_0 = (\{\emptyset\}, \emptyset)$. For $n \geq 1$ conditional on G_{n-1}:

1. Let $P_v \sim \mathrm{Poi}((1+p)(1-q)^{k_v})$ be independent for $v \in I_{n-1}$.
   - Form $\tilde{\mathcal{G}}_n$ from G_{n-1} by attaching P_v new vertices to v.
   - Let $\tilde{I}_n$ be the set of the new vertices.

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   - Let $\tilde{I}_n$ be the set of the new vertices.
2. Let $B_{\{v,w\}} \sim \mathrm{Ber}(q)$ be independent for each pair $v, w \in \tilde{I}_n$.
   - Let $v \sim w$ if $d_{\tilde{\mathcal{G}}_n}(v,w) = 4$ and $B_{\{v,w\}} = 1$.
   - Form G_n from $\tilde{\mathcal{G}}_n$ by identifying each class and merging multi-edges.

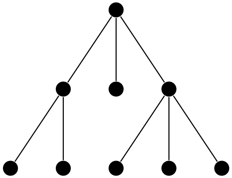
Example: Deletion and merger stages

Offspring: $\mathrm{Poi}((1+p)(1-q)^{k_v})$; Cousin mergers: $\mathrm{Ber}(q)$.

I_0 :

I_1 :

$\tilde{I}_2$:



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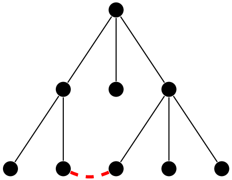

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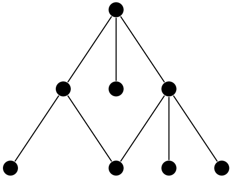

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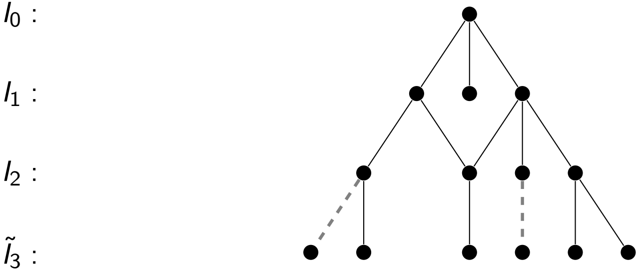

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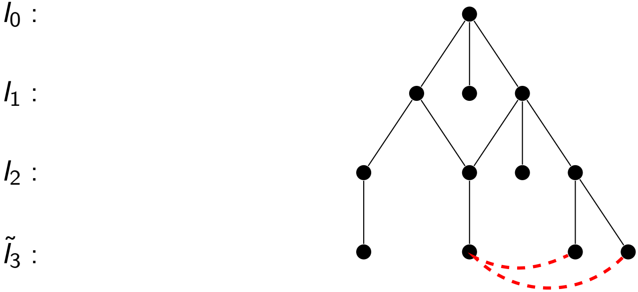

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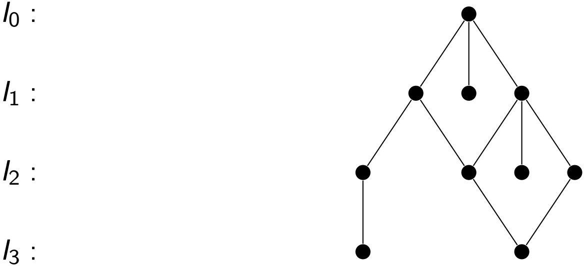

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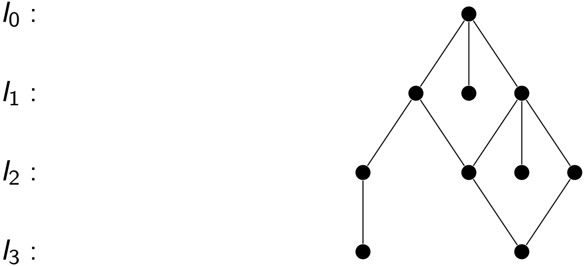


Difficulties:

- The process $(|I_n|, n \geq 0)$ is non-Markovian.
- Genealogical structure becomes a graph.
- Siblings may have distinct sets of ancestors.

Survival and extinction conditions

Offspring: $\mathrm{Poi}((1+p)(1-q)^{k_v})$; Cousin mergers: $\mathrm{Ber}(q)$.

Theorem (EPS, 21^+)

There is $C > 0$ and $p_0 \in (0,1)$ such that for $0 < p \leq p_0$

- *if $q < \frac{2}{5}p(1-Cp)$ then $\mathcal{G}(p,q)$ survives with positive probability;*
- *if $q > \frac{2}{5}p(1+Cp)$ then $\mathcal{G}(p,q)$ dies out almost surely.*

Open Question:

Is there a threshold $q_c(p)$ that determines the extinction of $G(p,q)$?

Open Question:

Is the survival probability of $G(p,q)$ monotone in p or q?

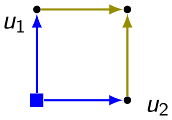
Combinatorial heuristic for 5/2 coefficient

The estimated average growth per generation is

$$\mathbb{E}[|I_n|] \approx \left(1 + p - \frac{5}{2} q\right) \mathbb{E}[|I_{n-1}|].$$

Pairs of cousins may be counted by their common grandparent:

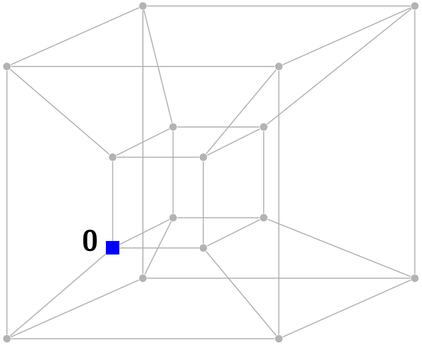
$$\begin{aligned}\mathbb{E}[\#\text{ pairs of cousins per grandparent}] &= \mathbb{E}[\#\text{ pairs of siblings}](1+p)^2 \\ &= \frac{(1+p)^4}{2}.\end{aligned}$$



$$\text{mergers} \approx \frac{1}{2} q (1+p)^4 \mathbb{E}|I_{n-2}|,$$

Per generation, there are deletions occurring 4 times as often!

Example: Exploration of random cluster



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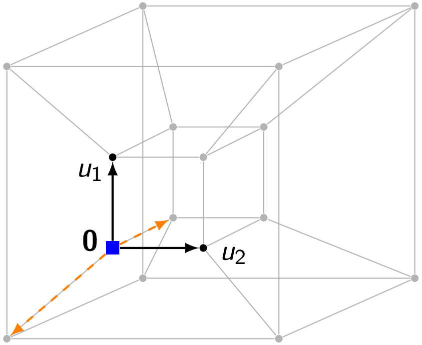

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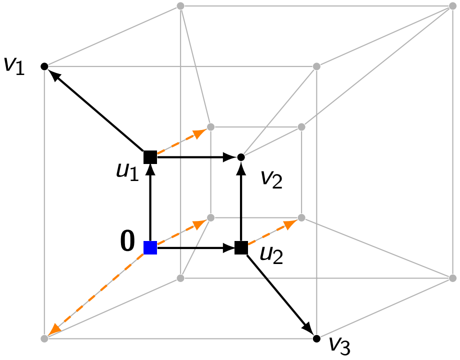

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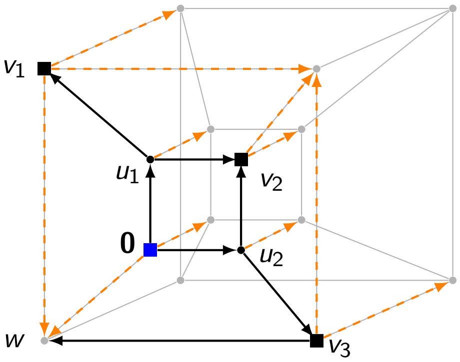

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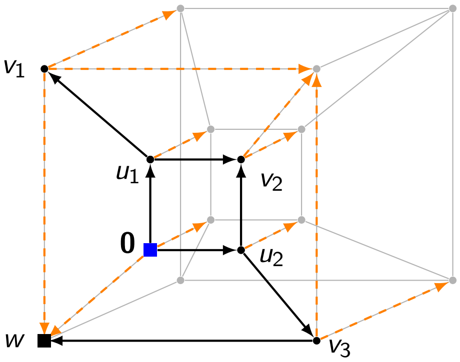

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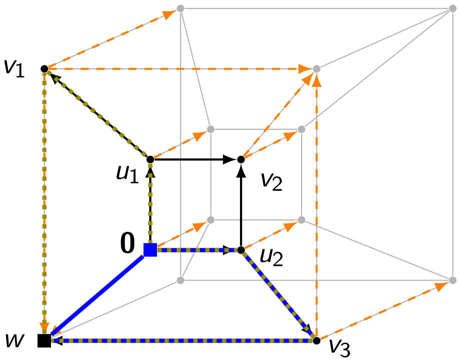


Detour: Let $u_1, \dots u_5$ be a non-backtracking walk in $\mathbb{G}$ then

$$c_b = \mathrm{P}(\text{walk forms a 4-cycle}) = \begin{cases} (\Omega - 1)^{-2} & \text{if } \mathbb{G} = Q_d \\ (\Omega - 1)^{-2} - (\Omega - 1)^{-3} & \text{if } \mathbb{G} = \mathbb{Z}_d \end{cases}$$

A non-backtracking walk $u_1, \dots u_{2\ell+1}$ forms a 2ℓ-cycle with probability $O(\Omega^{-\ell})$.

Exploration setup

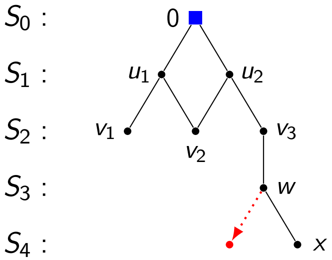
Let H be the percolation cluster of $\mathbf{0}$ in $\mathbb{G}_\rho$. The exploration process $H_n = (V_n, E_n)$, $n \geq 0$ is performed as follows: let $H_0 = (\{\mathbf{0}\}, \emptyset)$ and $T_{-1} = \emptyset$.

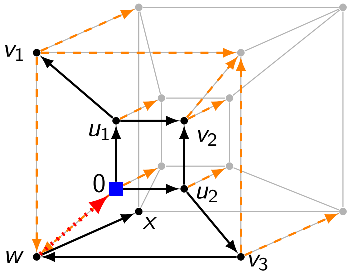
For $n \geq 0$, let $S_n = V_n \setminus V_{n-1}$ and

$$T_n = \{uv \in E(\mathbb{G}) : u \in V_n, v \notin V_n\},$$

and let F_n contain each $e \in T_n \setminus T_{n-1}$ independently with probability ρ. Then

$$E_n \setminus E_{n-1} = F_n \setminus T_{n-1}$$





Local properties of Q_d and $\mathbb{Z}^d$

Properties of $\mathbb{G}$ and exploration:

1. Edges only connect vertices in consecutive generations (no odd cycles)
2. Each $v \in S_{n-1}$ generates $\mathrm{Bin}(\Omega - \deg_{H_{n-1}}(v), \rho)$ edges in F_n (transitivity)
3. Deletion stage: Edges in $F_n \cap T_{n-1}$ were previously explored.
4. Merger stage: Edges in $E_n \setminus E_{n-1}$ may close cycles in H.

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Idealization of the exploration, for Ω large, using the probability that a non-backtracking walk forms a cycle:

Pairwise mergers: $\mathrm{Ber}(c_b(1 + o(1))$ $\sim \mathrm{Ber}((\Omega - 1)^{-2})$

* Neglecting correlations and cycles of length at least 6.

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$$\begin{aligned}
\text{Pairwise mergers:}\quad & \mathrm{Ber}(c_b(1+o(1))) && \sim \mathrm{Ber}((\Omega-1)^{-2}) \\
\text{Offspring:}\quad & \mathrm{Bin}(\Omega - \deg_{H_{n-1}}(v), \rho) && \sim \mathrm{Poi}((\Omega-1)\rho) \\
\text{Thinning factor:}\quad & \mathbf{1}\{vw \in F_n \setminus T_{n-1} \mid vw \in F_n\} && \sim \mathrm{Ber}(1-c_b)^{k_v}
\end{aligned}$$

* Neglecting correlations and cycles of length at least 6.

A special regime of our theorem

Consider $Q_{m+1,\rho}$ or $\mathbb{Z}_\rho^{(m+1)/2}$:

Exploration of a cluster is approximated by $\mathcal{G}(p(\rho), Q_b)$ where

$$1 + p(\rho) = m\rho, \quad q_b = m^{-2}.$$

Corollary (EPS, 21^+)

There is $K > 0$, such that under suitable conditions on ρ and m, letting

$$\hat{\rho}_c := m^{-1} + \frac{5}{2} m^{-3}.$$

- if $\rho > \hat{\rho}_c + Km^{-5}$ then $\mathcal{G}(p(\rho), q_b)$ survives with positive probability;
- if $\rho < \hat{\rho}_c - Km^{-5}$ then $\mathcal{G}(p(\rho), q_b)$ dies out almost surely.

Proof of Corollary: extinction phase

$$p(\rho) = m\rho - 1, \quad q_b = m^{-2} \quad \text{and} \quad \hat{\rho}_c := m^{-1} + \frac{5}{2}m^{-3}.$$

Let $\rho = \hat{\rho}_c - xm^{-5}$. Using that $p(\rho) = \frac{5}{2}m^{-2} - xm^{-4}$.

$$\begin{aligned} m^{-2} &\overset{?}{>} \frac{2}{5}p(\rho)(1 + Cp(\rho)) \\ &= \frac{2}{5}\left(\frac{5}{2}m^{-2} - xm^{-4}\right)\left(1 + \frac{5C}{2}m^{-2} - xCm^{-4}\right) \\ &= m^{-2} + \frac{2}{5}m^{-4}\left(Cx^2m^{-4} - x(1 + 5Cm^{-2}) + \frac{25}{4}C\right). \end{aligned}$$

We require that $Cx^2m^{-4} - x(1 + 5Cm^{-2}) + \frac{25}{4}C < 0$; that is, that $x \in (x_1, x_2)$ where x_1 and x_2 are the solutions to the quadratic equation.

If $x > K$ then for m large, $x \in (x_1, x_2)$ and $\mathcal{G}(p(\rho), q_b)$ dies out a.s.

Other critical probabilities

Borgs et al ['05], Federico et al. ['20], Heydenreich, Matzke ['19,'20]

For site percolation on $\mathbb{Z}^d$, as $d \to \infty$

$$p_c^s(\mathbb{Z}^d) = \Omega^{-1} + \frac{5}{2}\Omega^{-2} + \frac{31}{4}\Omega^{-3} + O(\Omega^{-4}).$$

Hamming graphs $H(d, m)$ have vertex set $\{0, 1, \dots, m-1\}^d$, edges join nearest neighbours. Let $\Omega = d(m-1)$ and $V = m^d$.

For $d \geq 2$, as $m \to \infty$

$$p_c(H(d, m)) = \Omega^{-1} + \frac{2d^2 - 1}{2(d-1)^2}\Omega^{-2} + O(\Omega^{-3}) + O(\Omega^{-1}V^{-1/3})$$

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Some differences with our heuristics:

- For site percolation, edges are not independently present.
- Hamming graphs contain shorter, odd cycles (triangles).

Summary

Exploration of a cluster of $Q_{m+1,\rho}$ or $\mathbb{Z}_\rho^{(m+1)/2}$ is approximated by $\mathcal{G}(p(\rho), Q_b)$.

$$p(\rho) = m\rho - 1, \quad q_b = m^{-2} \quad \text{and} \quad \hat{\rho}_c := m^{-1} + \frac{5}{2}m^{-3}.$$

Corollary (EPS, 21$^+$)

There is $K > 0$, such that under suitable conditions on ρ and m,

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