Survival for a Galton-Watson tree with cousin mergers

to approximate hypercube's percolation

Laura Eslava, Sarah Penington and Fiona Skerman

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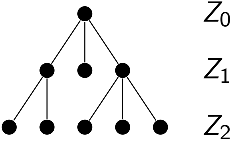
Poisson Galton-Watson trees $(p > -1)$

$P_v = \#$ offspring of v independent variables $\mathrm{Poi}(1+p)$.

Expected growth: Z_n is the nth generation size.



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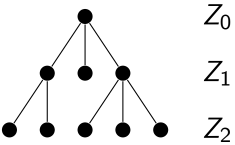

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Survival Threshold [G.,W., 1875]

For a Galton-Watson tree, if P_v satisfies $\mathbb{E}[P_v] = (1+p)$, then

- if $p > 0$ then the process survives with positive probability;
- if $p \leq 0$ then the process dies out with probability one.

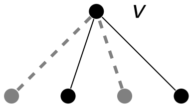
Proof Sketch: If $p < 0$ then

$$\mathrm{P}(Z_n \geq 1 \text{ for all } n) \leq P(Z_n \geq 1) \leq \mathbb{E}[Z_n] = (1+p)^n \to 0.$$

Slowing down the growth

One might trim each new generation by

- deleting some of the offspring, or
- identifying some individuals.



Independent offspring deletions:

Let $\mathcal{G}_d(p, q)$ be a population process where $P_v \sim \mathrm{Poi}(1 + p)$ and each of the new individuals is deleted with probability q.

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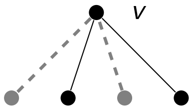

Independent offspring deletions:

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Theorem

For $p > 0$ and $q \in (0,1)$,

- *if $q < \frac{p}{1+p}$ then $\mathcal{G}_d(p,q)$ survives with positive probability;*
- *if $q \geq \frac{p}{1+p}$ then $\mathcal{G}_d(p,q)$ dies out with probability one.*

Proof. Per vertex, offspring distribution after deletions is $\mathrm{Poi}((1+p)(1-q))$. Then $\mathcal{G}_d(p,q)$ remains a Galton-Watson tree!

GW tree with cousin merges

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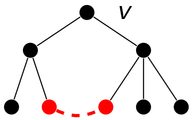

Independent cousin mergers:

Let $\mathcal{G}_c(p, q)$ be a population process where $P_v \sim \mathrm{Poi}(1 + p)$ and each pair of cousins is identified with probability q.

Difficulties:

- The process $(Z_n, n \geq 0)$ is non-Markovian.
- Genealogical structure becomes a graph.
- Siblings may have distinct sets of ancestors.

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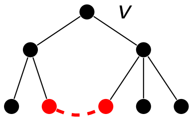

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Let $\mathcal{G}_c(p, q)$ be a population process where $P_v \sim \mathrm{Poi}(1 + p)$ and each pair of cousins is identified with probability q.

Theorem (E., Penington, Skerman, LAGOS 2021)

There is $C > 0$ and $p_0 \in (0, 1)$ such that for $0 < p \leq p_0$,

- *if $q < 2p(1 - Cp)$ then $\mathcal{G}_c(p, q)$ survives with positive probability;*
- *if $q > 2p(1 + Cp)$ then $\mathcal{G}_c(p, q)$ dies out with probability one.*

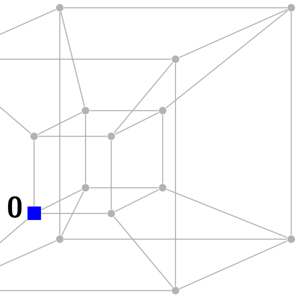
Resemblance to percolation?

Percolated graph G_ρ keeps edges independently with probability ρ.

Cluster exploration:

At each step,
traverse all edges of the cluster that are incident to discovered vertices.

Exploring cluster of $Q_{4,\rho}$



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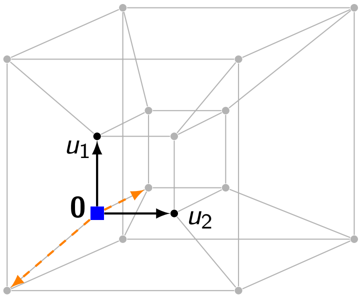

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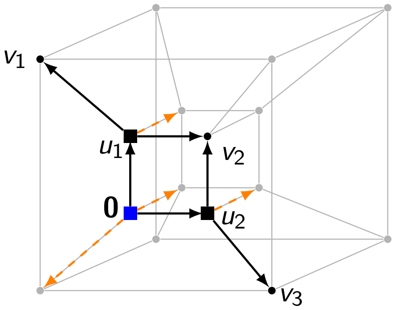

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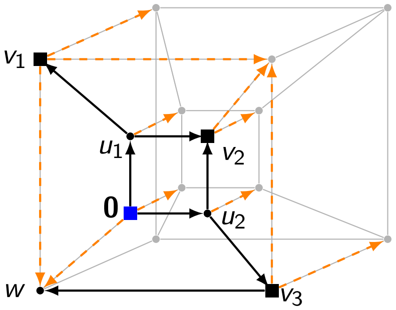

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- Edges may merge;
- some edges had been discarded.

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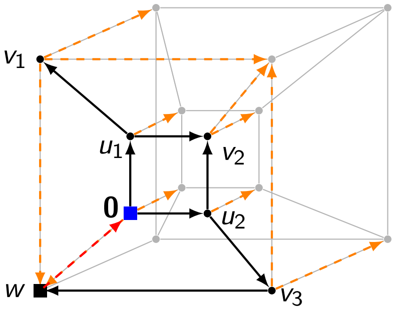
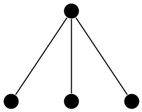


GW tree with deletions and cousin merges

For $p > -1$ and $q \in (0,1)$, define $\mathcal{G}(p,q)$ as follows.

Construction:



From nth to $(n+1)$th generation:

1. Individuals have $\mathrm{Poi}(1+p)$ offspring.
2. Deletions: each offspring of v survives independently with probability $(1-q)^{k_v}$.
3. Mergers: identify pairs of cousins independently with probablity q.

$k_v = \#\{\text{Individuals in previous generations at distance three}\}.$

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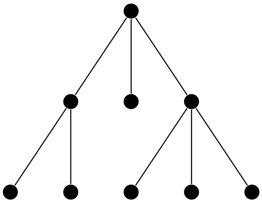

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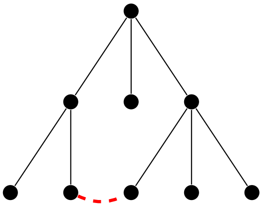

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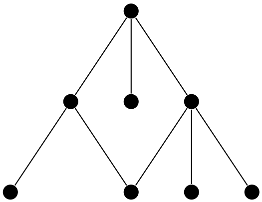

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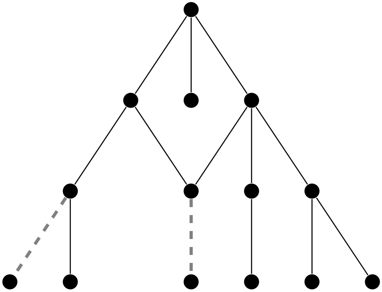

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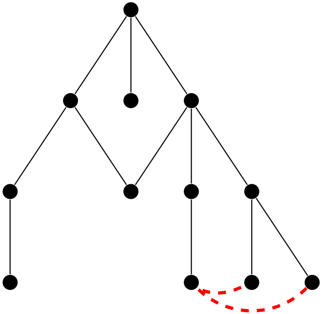

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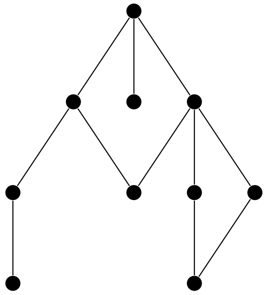

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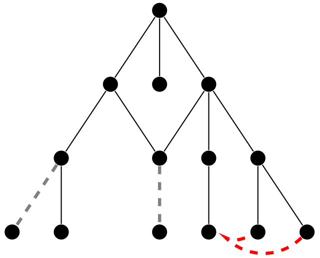
Survival threshold for $\mathcal{G}(p, q)$



- Offspring after **deletions**: $\mathrm{Poi}((1+p)(1-q)^{k_v})$
- Pairwise cousin mergers: $\mathrm{Ber}(q)$.

Theorem (E., Penington, Skerman, 21^+)

There is $C > 0$ and $p_0 \in (0,1)$ such that for $0 < p \le p_0$,

- *if $q < \frac{2}{5}p(1 - Cp)$ then $\mathcal{G}(p, q)$ survives with positive probability;*
- *if $q > \frac{2}{5}p(1 + Cp)$ then $\mathcal{G}(p, q)$ dies out with probability one.*

* See arxiv:2104.04407

The third model's the charm

Exploration of a cluster of $Q_{m+1,\rho}$ or $\mathbb{Z}_\rho^{(m+1)/2}$ is approximated by $\mathcal{G}(p(\rho), q_b)$.

$$p(\rho) = m\rho - 1, \quad q_b = m^{-2} \quad \text{and} \quad \hat{\rho}_c := m^{-1} + \frac{5}{2} m^{-3}.$$

Corollary (E., Penington, Skerman, 21^+)

There is $K > 0$, such that under suitable conditions on ρ and m sufficiently large,

- if $\rho > \hat{\rho}_c + Km^{-5}$ then $\mathcal{G}(p(\rho), q_b)$ survives with positive probability;
- if $\rho < \hat{\rho}_c - Km^{-5}$ then $\mathcal{G}(p(\rho), q_b)$ dies out with probability one.