Estadística bayesiana
y aplicaciones en ciencia de datos

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1. Statistical induction
2. Dirichlet process: the canonical BNP prior
3. BNP mixtures

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2. Dirichlet process: the canonical BNP prior

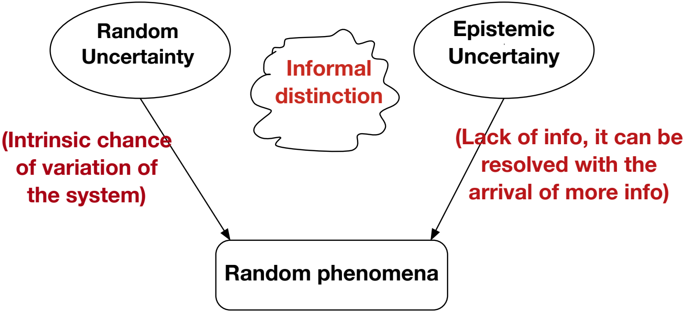
3. BNP mixtures

Random phenomena drive many aspects of this world

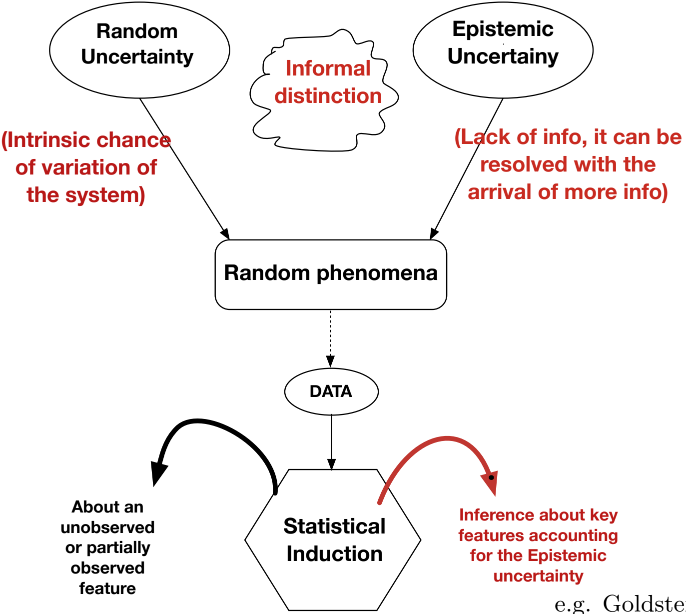
The Bayesian approach to statistical induction

Random phenomena

The Bayesian approach to statistical induction



The Bayesian approach to statistical induction



e.g. Goldstein (2013)

The basic probabilistic setup

- $(\Omega, \mathcal{A}, \mathbb{P})$: ***Probability space***
  - Ω –***sample space.*** Set of all possible outcomes
  - $\mathcal{A}$ –σ***-field.*** Collection of subsets of Ω with all events of interest
  - $\mathbb{P} : \mathcal{A} \mapsto [0, 1]$ –***Probability measure.*** Mathematically coherent measure to quantify all ***events*** $A \in \mathcal{A}$
- Features of interest can be translated into "numeric" quantities via
  - $(\mathbb{X}, \mathcal{X})$-valued functions, $X : \Omega \mapsto \mathbb{X}$. ***random variables*** (r.v.'s)
- Given a r.v. X, the set function defined by

$$\mathsf{P}_X(B) = \mathbb{P}(X^{-1}(B)), \qquad \text{for all } B \in \mathcal{X} \tag{1}$$

is termed the ***distribution or law*** of the random variable X.

  - When $\mathbb{X} = \mathbb{R}$ and $B = (-\infty, x]$ we write

$$\mathsf{F}_X(x) = \mathsf{P}_X((-\infty, x]) = \mathbb{P}(X \le x) \qquad \to \text{the } \textbf{\textit{(cdf)}} \text{ of } X$$

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Ex. toss a coin

$\Omega = \{\text{head}, \text{tail}\} = \{\omega_1, \omega_2\} = \{0, 1\}$

$\mathcal{A} = \{\Omega, \{0\}, \{1\}, \emptyset\}$

Let X the r.v. that assigns 1 if the outcome is tail and 0 otherwise, i.e. $\mathsf{P}_X(\{1\}) = \mathbb{P}(X(\omega_1) = 1)$ with $\mathbb{X} = \{0, 1\}$

- For such quantity, we might assign a value $\theta \in [0, 1]$, i.e.

$$\mathsf{P}_X(\{1\}) = \theta$$

$\Rightarrow$ Uncertainty about X is transferred to the ***parameter*** of interest θ.

How can we improve our knowledge about θ in the presence of observations from the random phenomena?

The basic setup

- Availability of more info about a random phenomenon

  ⇒ better uncertainty quantification

  ⇒ better statistical induction

- Realizations of a given phenomenon encoded via r.v.'s $\{X_i\}_{i\in\mathcal{I}}$
  - ▷ Logical/physical independence $\not\Rightarrow$ stochastic independence

    so $\mathbb{P}(X_{n+1} \in B \mid X_1, \ldots, X_n) = \mathbb{P}(X_{n+1} \in B)$ not always a good idea!
  - ▷ Statistical learning requires stochastic dependence !

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- Symmetry/Stability principles in the law modelling $\{X_i\}$'s are fundamental for statistical induction

  ▷ e.g. the past and future have similar behaviour

- Major symmetries used in statistics

  ▷ IID r.v.'s: physical & stochastic independence (rare in real apps!)

  ▷ Exchangeability: physical indep. + sampling order invariace!

  ▷ Stationarity: Uncertainty is not "time" invariant

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Exchangeable sequences

A finite sequence of r.v.'s, $\{X_n\}_{i=1}^{n}$, is said to be ***finite exchangeable*** if, for any permutation π of $(1, \ldots, n)$

$$(X_1, \ldots, X_n) \overset{\mathrm{d}}{=} (X_{\pi(1)}, \ldots, X_{\pi(n)})$$

An infinite sequence $\{X_n\}_{i=1}^{\infty}$ is said to be ***exchangeable*** if every subcollection is exchangeable.

- ≈ Distributional invariance under sampling order
- ▷ What can we say about the law of an exchangeable sequence

▷ B. de Finetti's representation characterises exchangeable sequences

de Finetti's representation Theorem: $\mathbb{X} = \{0, 1\}$ case

- B. de Finetti 1931: A seq. of binary r.v.'s $\{X_i\}_{i=1}^{\infty}$, *e.g.* with values in $\mathbb{X} = \{0, 1\}$, is exchangeable iff there exists a dist. q on $[0, 1]$

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \int_{[0,1]} \theta^{s_n}(1-\theta)^{n-s_n} \mathsf{q}(\mathrm{d}\theta)$$

where $s_n := \sum_{i=1}^{n} x_i$.

- $\mathsf{q}(\cdot)$ is the distribution of $\lim_{n\to\infty} \frac{s_n}{n}$
- Conditional independence

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n \mid \theta) = \prod_{i=1}^{n} \mathbb{P}(X_i = x_i \mid \theta) = \theta^{s_n}(1-\theta)^{n-s_n}$$

- This decomposition in "conditionally independent sample" given a random parameter "θ" justifies the Bayesian approach

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Example: $\{X_i\}_{i=1}^{\infty}$ be Bernoulli r.v.'s

Two different Bernoulli exchangeable laws by two different persons

$$\mathbb{P}(x_1,\ldots,x_n) = \frac{12}{s_n+2}\frac{1}{\binom{n+4}{s_n+2}} \quad \text{and} \quad \mathbb{P}(x_1,\ldots,x_n) = \frac{1}{[n+1]\binom{n}{s_n}},$$

- These persons believe that $\mathbb{P}(X_1 = 1) = 0.4$ & $\mathbb{P}(X_1 = 1) = 0.5$ resp.
- Both believe that $\Theta := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i$ exists & $\mathbb{P}(X_1 = 1 \mid \Theta = \theta) = \theta$
- Since $\mathbb{P}(X_1 = 1) = \mathbb{E}(\Theta)$ they must have different values for $\mathbb{E}(\Theta)$
- Assume we observe the result of $n = 20$ given by 14 "1s" and 6 "0s".

$$\mathbb{P}[X_{21} \mid x_1,\ldots,x_{20}] = 0.64 \quad \text{and} \quad \mathbb{P}[X_{21} \mid x_1,\ldots,x_{20}] = 0.68.$$

• Regardless of the prior mean on Θ, they should modify their opinion about the prop. of 1's!

• Consequence due to exchangeability, regardless of frequencies being interpreted as probabilities.

Schervish, 1995.

In Bayesian terms

If X is a r.v. on $\mathbb{X} = \{1, 2\}$ with prob. p_1 and $p_2 = 1 - p_1$ assigned to each element of $\mathbb{X}$. That is $\{X \mid p_1, p_2\} \sim \text{Bernoulli}(p_1, p_2)$

- Each value of $\mathbf{p} = (p_1, p_2)$ defines probability measure on $\mathbb{X}$
- $q = \mathsf{Be}(\alpha_1, \alpha_2)$ defines a probability measure on

$\mathcal{P}_{\mathbb{X}} :=$ {Space of prob. measures on $\mathbb{X}$}$=\{(p_1, p_2); p_i \geq 0 \text{ y } p_1 + p_2 = 1\}$

$$\Rightarrow \quad p_1 \mid X^{(n)} \sim \mathsf{Be}\left(\alpha_1 + \sum_{i=1}^{n} \delta_{X_i}(\{1\}), \alpha_2 + \sum_{i=1}^{n} \delta_{X_i}(\{2\})\right)$$

▷ If $\mathbb{X} = \{1, 2, \ldots, K\}$ then $X \mid (p_1, p_2, \ldots, p_k) \sim \prod_{i=1}^{k} p_i^{\delta_X(\{i\})}$

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One parameter per value in the support $\mathbb{X}$!

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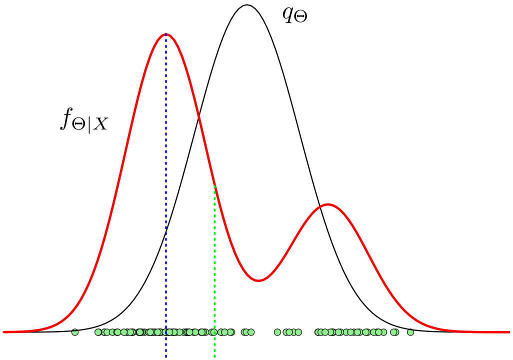
Prior to posterior effect



Exchangeable sequences: general $\mathbb{X}$

- Let $\mathcal{P}_{\mathbb{X}}$ be the space of all probability measures on $(\mathbb{X}, \mathcal{X})$

A seq. $\{X_i\}_{i=1}^{\infty}$ is exchangeable iff there exists Q on $\mathcal{P}_{\mathbb{X}}$ such that

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \int_{\mathcal{P}_{\mathbb{X}}} \prod_{i=1}^{n} \mathsf{P}(A_i)\, \mathsf{Q}(d\mathsf{P}), \quad \forall n \geq 1 \text{ and } A_i \in \mathcal{X}$$

Alternatively: $X_i \mid \mathsf{P} \overset{\text{iid}}{\sim} \mathsf{P}$ and $\mathsf{P} \sim \mathsf{Q}$ (conditionally iid) .

Hewitt and Savage 1955

- If $P_n(A) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(A)$ denotes the empirical dist. hence, Q is the dist. of the RPM P, where $\mathbb{P}[P_n \Rightarrow \mathsf{P}] = 1 \quad (\mathsf{P} \sim \mathsf{Q})$
- Q is unique
- "The unknown", P, that allows us to disaggregate the elements of $X^{(\infty)}$ as a conditional iid sample, is random.

Consequences of de Finetti's representation

- There is a clear bijection between the law of $\{X_i\}_{i=1}^{\infty}$ and Q
  - Pick $\mathsf{Q} \Rightarrow$ we have a law for $\{X_i\}_{i=1}^{\infty}$
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    i.e. $\mathsf{P}[X_{n+1} \in A \mid X^{(n)}] = \mathbb{E}_{\mathsf{Q}_{X^{(n)}}}[\mathsf{P}(A)]$ characterizes Q with $\mathsf{Q}_{X^{(n)}}(B) := \mathbb{P}(\mathsf{P} \in B \mid X^{(n)})$,
- However, any $\mathsf{P} \in \mathcal{P}_{\mathbb{X}}$ can be seen as the limit of P_n!
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  Q takes the interpretation of prior distributions on P

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    i.e. $\mathsf{P}[X_{n+1} \in A \mid X^{(n)}] = \mathbb{E}_{\mathsf{Q}_{X^{(n)}}}[\mathsf{P}(A)]$ characterizes Q with $\mathsf{Q}_{X^{(n)}}(B) := \mathbb{P}(\mathsf{P} \in B \mid X^{(n)})$,
- However, any $\mathsf{P} \in \mathcal{P}_{\mathbb{X}}$ can be seen as the limit of P_n!
- Bayesian interpretation:
  Q takes the interpretation of prior distributions on P

Probabilistically speaking the Bayesian approach is equivalent to the exchangeability assumption of the $\{X_i\}_{i=1}^{\infty}$

de Finetti and the Bayesian approach

The law of the exchangeable r.v's (and thus Q) is characterized by the conditional probabilities (or predictive distributions)

$$\mathbb{P}\left[X_{n+1} \in A_{n+1} \mid X_1 \in A_1, \ldots, X_n \in A_n\right] = \frac{\mathsf{E}_{\mathsf{Q}}\left[\prod_{i=1}^{n+1} \mathsf{P}(A_i)\right]}{\mathsf{E}_{\mathsf{Q}}\left[\prod_{i=1}^{n} \mathsf{P}(A_i)\right]} = \mathsf{E}_{\mathsf{Q}_{X^{(n)}}}\left[\mathsf{P}(A_{n+1})\right]$$

for all $n > 1$, with $P_0 := \mathbb{P}[X_1 \in A_1] = \mathsf{E}_{\mathsf{Q}}[\mathsf{P}(A_1)]$ and where

$$\mathsf{Q}_{X^{(n)}}(\mathrm{d}\mathsf{P}) = \frac{\prod_{i=1}^{n} \mathsf{P}(A_i)\, \mathsf{Q}(\mathrm{d}\mathsf{P})}{\mathsf{E}_{\mathsf{Q}}\left[\prod_{i=1}^{n} \mathsf{P}(A_i)\right]}, \quad \text{(dominated case)}$$

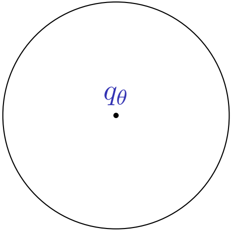
the posterior distribution of P given $X^{(n)} := (X_1, \ldots, X_n)$

Exchangeability: statistical learning for physically independent observations

Random phenomena encoded in $\mathbb{X}$-valued $\{X_i\}_{i=1}^{\infty}$ exchangeable sequence driven by $\mathsf{P} \sim \mathsf{Q}$

- $\mathsf{Q}(\cdot) = \delta_{q_\theta}(\cdot) \Rightarrow X_i$'s are iid

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \int_{\mathcal{P}_{\mathbb{X}}} \prod_{i=1}^{n} \mathsf{P}(A_i)\, \delta_{q_\theta}(d\mathsf{P}) = \prod_{i=1}^{n} q_\theta(A_i)$$



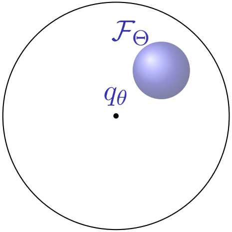
$\mathcal{P}_{\mathbb{X}}$: Space of all distributions on $\mathbb{X}$

Exchangeability: statistical learning for physically independent observations

Random phenomena encoded in $\mathbb{X}$-valued $\{X_i\}_{i=1}^{\infty}$ exchangeable sequence driven by $\mathsf{P} \sim \mathsf{Q}$

- $\mathsf{Q}(\mathcal{F}_\Theta) = 1 \Rightarrow$ Parametric family

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \int_{\mathcal{F}_\Theta} \prod_{i=1}^{n} \underbrace{F_\theta(A_i)}_{\text{Random uncertainty via param. model}} \overbrace{\pi_\theta(d\theta)}^{\text{Epistemic uncertainty}}$$



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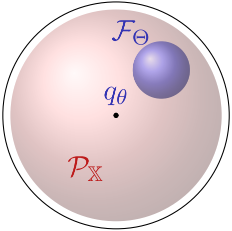
Exchangeability: statistical learning for physically independent observations

Random phenomena encoded in $\mathbb{X}$-valued $\{X_i\}_{i=1}^{\infty}$ exchangeable sequence driven by $\mathsf{P} \sim \mathsf{Q}$

- $\mathsf{Q}(\mathsf{P} : \mathsf{d}(\mathsf{P}, \eta) < \varepsilon) > 0, \ \forall\, \eta \in \mathcal{P}_{\mathbb{X}}$ y $\varepsilon > 0 \Rightarrow$ BNP

$$\mathbb{P}(X_1 \in A_1, \ldots, X_n \in A_n) = \int_{\mathcal{P}_{\mathbb{X}}} \prod_{i=1}^{n} \underbrace{\mathsf{P}(A_i)\, \mathsf{Q}(d\mathsf{P})}$$

Random and epistemic uncertainties in one stroke!



$\mathcal{P}_{\mathbb{X}}$: Space of all distributions on $\mathbb{X}$
…or other infinite dimensional sub-spaces of interest, $\mathcal{P}_{\mathbb{X}}^{d}$, $\mathcal{P}_{\mathbb{X}}^{c}$, etc.

Bayesian nonparametrics

What happens if $\mathbb{X}$ is of an infinite nature?

- We could $\mathcal{P}_{\mathbb{X}}\big|_{\mathcal{F}_\Theta}$, but doesn't resolve the "random uncertainty"
- We want models Q giving positive prob. to all elements of $\mathcal{P}_{\mathbb{X}}$, or at least some infinite subset, e.g. set of densities, cdf's, etc.
- de Finetti's representation Th. for general $\mathbb{X}$ gives an answer...
- Remember: $\{X_i\}_{i=n}^{\infty}$ exchangeable is driven by $\mathrm{P} \sim \mathrm{Q}$

How to construct suitable models for Q (nonparametric priors!)?

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How to construct suitable models for Q (nonparametric priors!)?

The Dirichlet distribution

Let $Z_i \stackrel{\text{iid}}{\sim} \mathsf{Ga}(\alpha_i, 1)$, $i = 1, \ldots, m$ and $\mathbf{W} := (W_1, \ldots, W_m)$ with

$$W_i = \frac{Z_i}{\sum_{i=1}^m Z_i}, \quad i = 1, \ldots, m \qquad \Rightarrow \mathbf{W} \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_m)$$

and is independent of $Z := \sum_{i=1}^m Z_i \sim \mathsf{Ga}(\sum_{i=1}^m \alpha_i, 1)$ with density

$$f(\mathbf{w}) = \frac{\Gamma\left(\sum_{i=1}^m \alpha_i\right)}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^{m-1} w_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^{n-1} w_i\right)^{\alpha_m - 1} \mathbb{I}_{\Delta_{m-1}}(\mathsf{w}),$$

where $\Delta_{m-1} := \left\{(w_1, \ldots, w_{m-1}) : w_i \geq 0, \sum_{i=1}^{m-1} w_i \leq 1\right\}$

Properties of Dirichlet distribution

Moments

Let $\alpha := \sum_{i=1}^{m} \alpha_i$ and $p_i := \alpha_i/\alpha$ hence

- $\mathsf{E}[w_i] = p_i$
- $\mathsf{Var}[w_i] = \frac{p_i(1-p_i)}{\alpha+1}$
- $\mathsf{Corr}[w_i, w_j] = -\frac{p_i p_j}{\alpha+1}$

Addition property

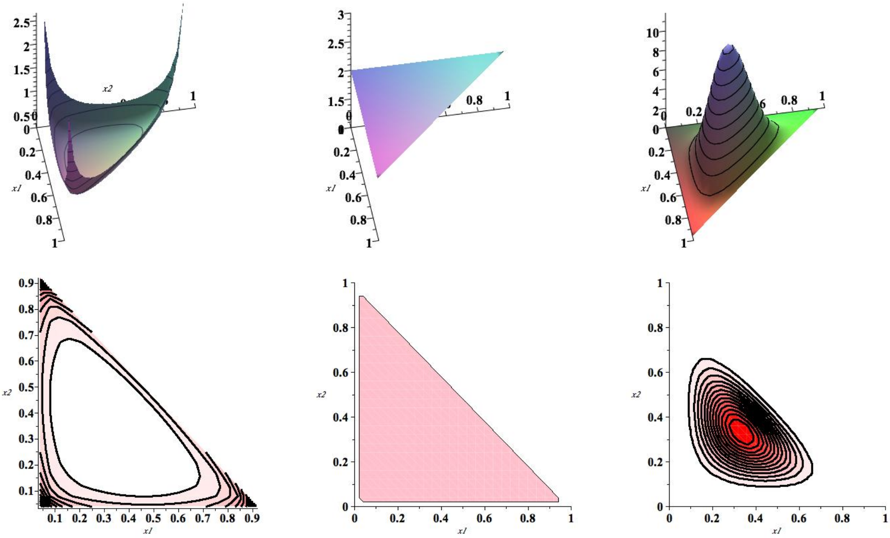
If W $\sim$ Dirichlet$(\alpha_1, \alpha_2, \ldots, \alpha_m)$ then

i) For any partition $A_1, \ldots, A_k$ of $\{1, \ldots, n\}$, the vector

$$\left(\sum_{i \in A_1} w_i, \sum_{i \in A_2} w_i, \ldots, \sum_{i \in A_k} w_i\right) \sim \text{Dirichlet}(\alpha'_1, \ldots, \alpha'_k)$$

where $\alpha'_i := \sum_{j \in A_i} \alpha_j$

$\alpha_1 = \alpha_2 = \alpha_3 = 0.2$ | $\alpha_1 = \alpha_2 = \alpha_3 = 1$ | $\alpha_1 = \alpha_2 = \alpha_3 = 5$



Ferguson 1973: The canonical example

1. Via infinite dimensional distributions with pre-scribed fdds

Let $\alpha > 0$ a non-atomic finite measure on a Polish space $(\mathbb{X}, \mathcal{X})$. A $\mathcal{P}_{\mathbb{X}}$-valued RPM, P, is said to have a Dirichlet process $(\mathcal{D}_\alpha)$ distribution, if for all measurable partition $(B_1, \ldots, B_k)$ de $\mathbb{X}$

$$(\mathsf{P}(B_1), \ldots, \mathsf{P}(B_k)) \sim \mathrm{Dir}(\alpha(B_1), \ldots, \alpha(B_k))$$

- Ferguson 73' proved that the Dirichlet dist. is projective and therefore Daniel-Kolmogorov's existence theorem ensures the existence of $\mathcal{D}_\alpha$. Namely, a stochastic process indexed on $\mathcal{X}$.

The Dirichlet process $\mathcal{D}_\alpha$: The canonical example

Extending the finite-dim properties to the infinite-dim object it can be seen that if $X_i \stackrel{\text{iid}}{\sim} \mathsf{P}$ and $\mathsf{P} \sim \mathcal{D}_\alpha$ then

- $P_0(B) := \mathsf{E}_{\mathcal{D}_\alpha}[\mathsf{P}] = \frac{\alpha(B)}{\theta}$ for $B \in \mathcal{X}$ and where $\theta := \alpha(\mathbb{X})$
- $\mathsf{Var}_{\mathcal{D}_\alpha}[\mathsf{P}(B)] = \frac{P_0(B)(1-P_0(B))}{\theta+1}$
- $\mathsf{Cov}(\mathsf{P}(B_1), \mathsf{P}(B_2)) = \frac{P_0(B_1 \cap B_2) - P_0(B_1)P_0(B_2)}{\theta+1}$

If $X_i \mid \mathsf{P} \stackrel{\text{iid}}{\sim} \mathsf{P}$ y $\mathsf{P} \sim \mathcal{D}_{\theta P_0}$, then $X_i \sim P_0, \; \forall \; i = 1, 2, \ldots$

$$\mathsf{P} \mid X_1, \ldots, X_n \sim \mathcal{D}_{\theta P_0 + n P_n} \quad \text{(conjugacy)}$$

$$\mathsf{E}[\mathsf{P} \mid X_1, \ldots, X_n] = \frac{\theta}{\theta+n} P_0 + \frac{n}{\theta+n} \sum_{i=1}^{n} \frac{\delta_{X_i}}{n}, \quad \text{(Bayes estimator)}$$

Ferguson (1973)

The Dirichlet process $\mathcal{D}_\alpha$: Pólya urn representation

2. Specification of Q via predictive distributions.

- Q can be characterized by its predictive dist. (Bayes estimator)

$$\mathbb{P}(X_{n+1} \in A \mid X_1, \ldots, X_n) = \mathsf{E}\left[\mathsf{P}(A) \mid X_1, \ldots, X_n\right] = \frac{\alpha_n(A)}{\alpha_n(\mathbb{X})}$$

with $\alpha_n(\cdot) = \alpha(\cdot) + \sum_{i=1}^n \delta_{X_i}(A)$. In other terms

$$\mathbb{P}[X_{n+1} \in \cdot \mid X^{(n)}] = \underbrace{\frac{\theta}{\theta + n}}_{\mathbb{P}[X_{n+1} = \text{“new”} \mid X^{(n)}]} \overbrace{P_0(\cdot)}^{\text{Prior guess}} + \underbrace{\frac{n}{\theta + n}}_{\mathbb{P}[X_{n+1} = \text{“old”} \mid X^{(n)}]} \overbrace{\sum_{i=1}^n \frac{\delta_{X_i}}{n}(\cdot)}^{\text{empirical measure}},$$

- Q is a DP iff the predictive is a linear combination of P_0 and the empirical measure

Regazzini (1978); Lo (1991)

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The Dirichlet process $\mathcal{D}_\alpha$: Pólya urn representation

Blackwell and MacQueen 73' observed that when $n \to \infty$

$$\frac{\alpha_n()}{\alpha_n(\mathbb{X})} \overset{\text{a.s.}}{\to} \mathsf{P}, \qquad \text{with} \qquad \mathsf{P} \sim \mathcal{D}_\alpha$$

→ Very appealing for MCMC implementations

→ A direct consequence is that

$$\mathbb{P}(X_i = X_j) = \frac{1}{\theta + 1} > 0, \qquad i \neq j$$

Blackwell 73' proved that

- $\mathcal{D}_\alpha(\mathrm{P} : \mathrm{P} \text{ is discrete }) = 1$

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