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CONTRIBUTIONS TO BRANCHING STRUCTURES IN RANDOM ENVIRONMENTS

T E S I S

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Abstract

This thesis focuses on Markovian random models with a branching structure which interacts with a random environment. These processes are widely used to model random phenomena in nature such as: the size of an asexual population, the expansion of a virus in a population, among others. In this work, we are not only interested in models of population dynamics and virus expansion, but also in how they interact with the environment. We are interested in how environment affects our original models.

This thesis is divided into three parts where our problems are framed. The first set of results focuses on Galton-Watson processes in a varying environment. Based on a two-spine decomposition technique, we provide a probabilistic argument of a Yaglom-type limit for this family of processes. The result states that, when the process is in the critical regime, a suitable normalisation of the process conditioned on non-extinction converges in distribution to a standard exponential random variable.

The second part deals with an inhomogeneous contact process on a Galton-Watson tree. The contact process is a simple model for the spread of an infection in a structured population. Here, we consider a variant of the contact process on Galton-Watson trees, where vertices are equipped with a random fitness which represents inhomogeneities among individuals. In particular, we establish conditions under which the contact process with fitness on Galton-Watson trees exhibits a phase transition. Furthermore, we show that if we start with a finite configuration of infected vertices then, almost surely, the configuration remains finite for all times.

In the last part of the thesis, we consider continuous state branching processes in a Lévy environment. Here, we are interested in understanding the asymptotic behaviour of the non-extinction and non-explosion probabilities for this family of processes. For the explosion problem, we assume that the branching mechanism corresponds to the negative of the Laplace exponent of a subordinator. We extend the characterisation of the quenched Laplace exponent and derive a necessary and sufficient condition for explosion. Further, the long-term behaviour of the non-explosion probability is studied in the critical and subcritical explosion regimes. On the other hand, we also study the speed of extinction of continuous state branching processes in subcritical Lévy

environments. More precisely, when the associated Lévy process to the environment drifts to minus infinity and, under a suitable exponential change of measure, the resulting process either drifts to minus infinity or oscillates.

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Introduction

This thesis focuses on Markovian probabilistic models that interact with a random environment. Markov processes are random processes that fulfill the property that, conditionally to the current value, the future and the past are independent. These processes are widely used to model random phenomena in nature such as: the size of an asexual population, the expansion of a virus in a population, among others. In this work, we are not only interested in models of population dynamics and virus expansion, but also in how they interact with the environment. We are interested in the knowledge of how environmental affect our original models. In other words, in how the dynamics of a population are affected by the environment throughout their reproductions laws or how the environment affects the dynamics of a virus in a given population. As we can see, the area in which we carry out this research has a strong biological motivation, especially in the areas of ecology and epidemiology. Here, we are interested in three probabilistic models that interact with a random environment. More precisely, this thesis is divided in three parts where our problems are framed.

- Part I: Galton-Watson processes in varying environment (Chapter 1).
- Part II: Contact processes with fitness on Galton-Watson trees (Chapter 2).
- Part III: Continuous-state branching processes in a Lévy environment (Chapters 3, 4 and 5).

Part I of this thesis is dedicated to the study of Yaglom's limit for critical Galton-Watson processes in varying environment. This part is a joint work with Sandra Palau and the results have been published in

[17] N. Cardona-Tobón and S. Palau. Yaglom's limit for critical Galton-Watson process in varying environment: A probabilistic approach. *Bernoulli*, 27(3):1643-1265, 2021.

Part II of this thesis deals with phase transitions for contact processes with fitness on Galton-Watson trees. This part is based on a joint work with Marcel Ortgiese and has resulted in the preprint

[19] N. Cardona-Tobón and M. Ortgiese. The contact process with fitness on Galton-Watson trees. *Preprint arXiv:2110.14537*, 2021.

Part III of this thesis is dedicated to the study of the speed of the probability of non-explosion and non-extinction for continuous-state branching process in a Lévy environment. This part is based in a joint collaboration with Juan Carlos Pardo. All the results here presented will be part of two forthcoming research papers one of them is the following preprint.

[18] N. Cardona-Tobón and J.C. Pardo. Speed of extinction for continuous state branching processes in subcritical Lévy environments: the strongly and intermediate regimes *Preprint arXiv:2112.13674*, 2021.

Below, we briefly introduce the models studied in this thesis and also review the results obtained during my PhD studies.

Part I: Galton-Watson processes in varying environment

A Galton Watson process models the size of a population in which, at every generation, each individual give births according to a fixed offspring distribution and independently of the other individuals. More precisely, a *Galton-Watson process* $Z = \{Z_n : n \geqslant 0\}$ is a discrete-time Markov chain which evolves according to the following recurrence formula

$$Z_0 = 1 \qquad \text{and} \qquad Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)},$$

where $\{\xi_i^{(n)}, i \geqslant 0\}$ forms a family of independent and identically distributed random variables with common distribution $(p_k, k \geqslant 0)$. For a complete overview of the theory of Galton-Watson process the reader is referred to the monograph of Athreya [5].

Let $m = \sum_{k=0}^{\infty} kp_k$ be the mean of the reproduction law. A Galton-Watson process is called *critical*, *subcritical* or *supercritical* accordingly as $m = 1$, $m < 1$ or $m > 1$, respectively. It is well-known that Galton-Watson processes become extinct with probability 1 if and only if $m \leqslant 1$ and $p_1 < 1$. Now, in order to describe the asymptotic behaviour of $(Z_n, n \geqslant 0)$ in the critical and subcritical regimes, Yaglom [75] studied the process Z conditioned on the event of non-extinction, i.e. on $\{Z_n > 0\}$. When $m < 1$ and under a certain moment restriction, Yaglom proved that the law of $\{Z_n | Z_n > 0\}$ converges to a proper distribution. Nevertheless, in the case of $m = 1$, which is in some sense the most interesting regime, the process $\{Z_n, | Z_n > 0\}$ diverges to ∞. Thus, a suitable normalization is needed to make the conditioned process to converge to a positive non-degenerated limit. In other words, Yaglom's theorem states that if $m = 1$ and $\sigma^2 := \sum_{k=0}^{\infty}(k-1)p_k < \infty$, we have

$$\left(\frac{Z_n}{n}; \mathbb{P}(\,\cdot\,| Z_n > 0)\right) \xrightarrow{(d)} (Y; \mathbb{P}), \qquad \text{as } \ n \to \infty,$$

where Y is an exponential random variable with mean $\sigma^2/2$ (see for instance Kesten et al. [46]). Under a third moment assumption, this result is originally due to Yaglom [75]. Further, in the literature, we can find different proofs of Yaglom's theorem, see e.g. Lyons et al. [55] and Geiger [31, 32]. More recently, Ren et al. [67], developed another new proof using a two-spine decomposition technique.

By introducing a varying environment in the model, the reproduction laws will change from generation to generation. In other words, different individuals give birth independently and their offspring distributions coincide within each generation but vary among generations. For example, think of a population having one year life cycle. Each year, the weather conditions or the resource supply (the environment) vary, which influences the reproductive success of the population. More precisely, for a *varying environment* $Q = (q_1, q_2, \ldots)$ of probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, a *Galton-Watson process* $Z^Q = \{Z_n^Q : n \geqslant 0\}$ *in a varying environment* Q (GWVE for short) is a Markov chain defined recursively as follows

$$Z_0^Q = 1 \quad \text{ and } \quad Z_n^Q = \sum_{i=1}^{Z_{n-1}^Q} \chi_i^{(n)}, \qquad n \geqslant 1,$$

where $\{\chi_i^{(n)} : i, n \geqslant 1\}$ is a sequence of independent random variables such that

$$\mathbb{P}(\chi_i^{(n)} = k) = q_n(k), \qquad k \in \mathbb{N}_0, \ i, \ n \geqslant 1.$$

The variable $\chi_i^{(n)}$ denotes the offspring of the i-th individual in the $(n-1)$-th generation. For a GWVE, we have a new classification which extends the one already mentioned for classical Galton-Watson processes. To be more precise, denote by η the probability of extinction, i.e $\eta := \mathbb{P}(Z_n^Q = 0 \text{ for some } n)$. Here, we exclude some processes that will have unusual behaviour and we work with regular processes. In other words, according with Kersting, [44], we say that a GWVE is *regular* if there exists a constant $c > 0$ such that for all $n \geqslant 1$,

$$\mathbb{E}\left[(\chi_i^{(n)})^2 \mathbf{1}_{\{\chi_i^{(n)} \geqslant 2\}}\right] \leqslant c\mathbb{E}\left[\chi_i^{(n)} \mathbf{1}_{\{\chi_i^{(n)} \geqslant 2\}}\right] \mathbb{E}\left[\chi_i^{(n)} \middle| \chi_i^{(n)} \geqslant 1\right].$$

The latter regularity assumption is considerably mild. As we will explain in the next chapter. Now, we say that a regular GWVE is

- *supercritial* if $\eta < 1$ and $\lim_{n\to\infty} \mathbb{E}[Z_n^Q] = \infty$,
- *asymptotically degenerate* if $\eta < 1$ and $\lim_{n\to\infty} \mathbb{E}[Z_n^Q] < \infty$,
- *critical* if $\eta = 1$ and $\lim_{n\to\infty} \mathbb{E}[Z_n^Q | Z_n^Q > 0] = \infty$,
- *subcritical* if $\eta = 1$ and $\lim_{n\to\infty} \mathbb{E}[Z_n^Q | Z_n^Q > 0] < \infty$.

For further details about GWVEs, we refer to the monograph of Kersting and Vatutin [45].

The first part of this thesis concerns with Yaglom's theorem for Galton-Watson processes in varying environment. In order to present our main result we first need to introduce additional notation. Denote by $\{f_n, n \geqslant 1\}$ the corresponding sequence of generating functions of $\{\chi_i^{(n)} : i, n \geqslant 1\}$, i.e. $f_n(s) := \mathbb{E}[s^{\chi_i^{(n)}}]$ for $s \in [0,1]$. Define the following two sequences,

$$\mathbb{E}[Z_n^Q] = \mu_n, \qquad \text{and} \qquad \frac{\mathbb{E}[Z_n^Q(Z_n^Q - 1)]}{\mathbb{E}[Z_n^Q]^2} = \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}, \qquad n \geqslant 1,$$

where $\mu_0 := 1$ and for any $n \geqslant 1$,

$$\mu_n := f_1'(1) \cdots f_n'(1), \qquad \text{and} \qquad \nu_n := \frac{f_n''(1)}{f_n'(1)^2}.$$

Actually these quantities essentially dictate the behaviour of GWVEs (see Kersting [44]). Given a varying environment Q, we define the sequence $\{a_n^Q : n \geqslant 0\}$ as follows

$$a_0^Q = 1, \qquad \text{and} \qquad a_n^Q = \frac{\mu_n}{2} \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}, \qquad n \geqslant 1.$$

Further, we assume the following condition

$$\text{there exists } c>0 \text{ such that } \ f_n'''(1) \leqslant c f_n''(1)(1+f_n'(1)), \quad \text{for any } n \geqslant 1. \tag{1}$$

We state now our main result of the first part of this thesis.

Theorema 0.0.1 (Yaglom's limit)**.** *Let* $\{Z_n^Q : n \geqslant 0\}$ *be a critical GWVE that satisfies condition* (1)*. Then*

$$\left(\frac{Z_n^Q}{a_n^Q}; \mathbb{P}(\,\cdot\,|Z_n^Q > 0)\right) \xrightarrow{(d)} (Y; \mathbb{P}), \qquad \textit{as} \ \ n \to \infty,$$

where Y *is a standard exponential random variable.*

In the literature, we can find different proofs of this result under stronger conditions than ours and/or using analytical techniques. Probably the first proof was carried out by Jagers [42]. He gave a proof of Yaglom's limit under certain extra assumptions. Afterwards, Bhattacharya and Perlman [14] presented a generalisation of Jager's result but with stronger conditions than the ones we consider here. Kersting [44] provided another proof in a similar framework than ours, that we will explain in more details in Chapter 1. Here, we provide a probabilistic proof of Yaglom's limit using the genealogies of the model rather than generating functions. Specifically, we use a two-spine decomposition technique, which was introduced by Ren et al. [67] to Yaglom's limit for classical Galton-Watson processes. In particular, our arguments allow us to give a more intuitive explanation of why the limit must have an exponential distribution and thus have a better understanding of the model.

Part II: Contact processes with fitness on Galton-Watson trees

The contact process is a classical interacting particle system modelling the spread of an infection in a given population. The population is modelled by a graph where vertices represent individuals susceptible to the infection and the edges, the connections between them. Given the infection parameter $\lambda > 0$, the dynamic of the contact process is as follows: at any time, each vertex of the graph is either infected or healthy. Each infected vertex infects each of its neighbours independently at rate λ and is healed at rate 1. Further, we assume that the infection and recovery events in the process happen independently.

The behaviour of the contact process depends on the infection parameter λ. In other words, depending on the value of λ, the process could present different phase transitions. For an infinite rooted graph, there are two critical values of interest $0 \leqslant \lambda_1 \leqslant \lambda_2$, which determine different regimes where the contact process exhibits extinction, weak survival or strong survival. In others words, in the *extinction* phase, for $\lambda \in (0, \lambda_1)$, the infection becomes extinct in finite time almost surely. In the *weak survival* phase, when $\lambda \in (\lambda_1, \lambda_2)$, the infection survives forever with positive probability, and the root is infected finitely many times almost surely. Finally, in the *strong survival* phase, for $\lambda \in (\lambda_2, \infty)$, the infection also survives forever with positive probability, however in this regime the root is infected infinitely many times with positive probability.

The phase diagrams of the contact processes on the integer lattice and on the infinite d-ary tree, are well-understood (see [37, 64, 69]). Recent progress has been done in more general graphs. Chatterjee and Durret [20] considered the contact process on models of power law random graphs and studied the critical value. Afterwards, Huang and Durret [41] and Bhamidi et al. [13] establish a necessary and sufficient criterion for the contact process on Galton-Watson trees to exhibit the phase of extinction.

A natural generalization of the contact process is to introduce inhomogeneity into the graph by associating a random fitness to each vertex that influences how likely the vertex is to receive and to pass on the infection. Peterson [66] introduced the contact process on a (deterministic) complete graph with random vertex-dependent infection rates. In this part of this thesis, we are interested in study the contact process with random weights in Galton-Watson trees. To present our main results, first we formally define the model. To this end, let us denote by $\mathcal{T} \sim \mathbf{GW}(\xi)$ the Galton-Watson tree rooted at ρ with offspring distribution $\mathscr{L}(\xi)$. We assume that $1 < \mathbb{E}[\xi] < \infty$. Denote by $V(\mathcal{T})$ the set of vertices in $\mathcal{T}$. We equip each vertex $v \in V(\mathcal{T})$ of the tree with a random initial fitness. More precisely, let $\mathbb{F}(\mathcal{T}) := (\mathcal{F}_v)_{v \in V(\mathcal{T})}$ be a sequence of i.i.d copies of a non-negative random variable. We denote by $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ this weighted version of the tree $\mathcal{T}$.

Definition 0.0.1. *Let $\mathbb{F}(\mathcal{T}) := (\mathcal{F}_v)_{v \in V(\mathcal{T})}$ be a sequence of i.i.d. copies of a random variable $\mathcal{F}$ taking values in $[1, \infty)$. The inhomogeneous contact process on $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ is a continuous-time Markov chain on the state space $\{0,1\}^{V(\mathcal{T})}$, where a vertex is either infected (state 1) or healthy (state 0). We denote the process by*

$$(X_t) \sim \mathbf{CP}\Big(\mathcal{T}; \mathbf{1}_A\Big),$$

where $\mathbf{1}_A$ *is the initial configuration in which the vertices in* $A \subset V(\mathcal{T})$ *are initially infected. Given the fitness* $\mathbb{F}(\mathcal{T})$ *and* $\lambda > 0$*, the process evolves according to the following rules.*

- *For each* $v \in V(\mathcal{T})$ *such that* $X_t(v) = 1$*, the process* X_t *becomes* $X_t - \mathbf{1}_v$ *with rate 1.*

- *For each* $v \in V(\mathcal{T})$ *such that* $X_t(v) = 0$*, the process* X_t *becomes* $X_t + \mathbf{1}_v$ *with rate*
$$\lambda \sum_{v \sim v'} \mathcal{F}_{v'} \mathcal{F}_v X_t(v'),$$
where $\mathcal{F}_v$ *and* $\mathcal{F}_{v'}$ *are the fitness associated to* v *and* v'*. The notation* $v \sim v'$ *means that vertices* v *and* v' *are connected by an edge in* $\mathcal{T}$*.*

Our first main result shows that if we start the inhomogeneous contact process on $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ with a finite configuration, then almost surely the configuration remains finite for all times.

Theorema 0.0.2. *Assume that* $\mu = \mathbb{E}[\xi] < \infty$*. Consider* $(X_t) \sim \mathbf{CP}(\mathcal{T}, \mathbf{1}_A)$ *the inhomogeneous contact process on* $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$*, started with any finite set* $A \subset V(\mathcal{T})$ *infected. Then*
$$\mathbb{P}\Big(|X_t| < \infty, \ \forall t \geqslant 0\Big) = 1.$$

Now, we define the critical values for the infection parameter λ. Given the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$, we define the threshold between extinction and weak survival by

$$\lambda_1(\mathcal{T}, \mathbb{F}(\mathcal{T})) := \inf \Big\{\lambda : \ \mathbb{P}\Big(X_t \neq \mathbf{1}_\emptyset \text{ for all } t \geqslant 0 \ \Big| \ \mathcal{T}, \mathbb{F}(\mathcal{T})\Big) > 0\Big\},$$

and the weak-strong survival threshold by

$$\lambda_2(\mathcal{T}, \mathbb{F}(\mathcal{T})) := \inf \Big\{\lambda : \ \liminf_{t \to \infty} \mathbb{P}\Big(\rho \in X_t \ \Big| \ \mathcal{T}, \mathbb{F}(\mathcal{T})\Big) > 0\Big\}.$$

Similar arguments as those used in Pemantle [64, Proposition 3.1], allow us to see that $\lambda_1(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ and $\lambda_2(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ are constant for almost every $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ conditioned on $|\mathcal{T}| = \infty$. We denote by λ_1 and λ_2 these two constants.

Our second main result tell us that if the distribution of the product of ξ and the fitness $\mathcal{F}$ has exponential tails, then the inhomogeneous contact process exhibits a phase of extinction.

Theorema 0.0.3. *Consider the inhomogeneous contact process on the tree* $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. *Suppose that only the root of the tree is initially infected. We assume that the distribution of the product of* ξ *and the fitness* $\mathcal{F}$ *has exponential tails, i.e.,*

$$\mathbb{E}\Big[e^{c\xi\mathcal{F}}\Big] = M < \infty \quad \textit{for some} \quad c, M > 0.$$

Then there exists $\lambda_0 > 0$ *such that for all* $\lambda < \lambda_0$, *the process dies out almost surely.*

Our last main result of this part of the thesis shows that, under a condition on either the fitness distribution or the offspring distribution, there is no phase transition and the process survives with positive probability for any choice of the infection parameter.

Theorema 0.0.4. *Consider the inhomogeneous contact process on the tree* $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. *Suppose that only the root of the tree is initially infected. Assume that* ξ *and* $\mathcal{F}$ *are unbounded and one of the following two conditions holds*

$$\limsup_{f\to\infty} \frac{\log \mathbb{P}(\mathcal{F} > f)}{\log f} = -C_1 \quad \textit{for some} \quad C_1 \in [0, \infty),$$

$$\limsup_{k\to\infty} \frac{\log \mathbb{P}(\xi = k)}{k} = -C_2, \quad \textit{for some} \quad C_2 \in [0, \infty),$$

Then $\lambda_1 = \lambda_2 = 0$, *i.e., the process survives strongly for any* $\lambda > 0$.

Part III: Continuous-state branching processes in a Lévy environment

Continuous state branching processes (CSBP for short) were introduced by Jirina [43] and since then, they have been studied by several authors. These processes allow to model the size of a sufficiently large population if it is assumed that environmental factors do not affect the population. More precisely, a CSBP is a $[0, \infty]$- valued strong Markov process $Y = \{Y_t, t \geqslant 0\}$ with probabilities $\{\mathbb{P}_x, x \geqslant 0\}$ such that, it has càdlàg paths and its law enjoys the branching property. That is, for all $\theta \geqslant 0$ and $x, y \geqslant 0$

$$\mathbb{E}_{x+y}\Big[e^{-\theta Y_t}\Big] = \mathbb{E}_x\Big[e^{-\theta Y_t}\Big]\mathbb{E}_y\Big[e^{-\theta Y_y}\Big].$$

Moreover, the law of Y_t is completely characterised by the latter identity, i.e.

$$\mathbb{E}_x\Big[e^{-\theta Y_t}\Big] = e^{-xu_t(\theta)},$$

where $u_t(\theta)$ is a differentiable function in t satisfying

$$\frac{\partial u_t}{\partial t}(\theta) + \psi(u_t(\theta)) = 0 \qquad \text{and} \qquad u_0(\theta) = \theta,$$

where the function ψ is called the *branching mechanism* and satisfies the celebrated Lévy-Khintchine formula,

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\right)\mu(\mathrm{d}x), \qquad \lambda \geqslant 0.$$

where $a \in \mathbb{R}$, $\gamma \geqslant 0$, and μ is a measure supported in $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty.$$

On the other hand, it is not difficult to deduce that

$$\mathbb{E}_x[Y_t] = xe^{-\psi'(0+)t}, \qquad x,\ t \geqslant 0. \tag{2}$$

This leads to a similar classification as in the discrete setting of Galton-Watson processes. More precisely, the process Y is called *supercritical* if $\psi'(0+) < 0$, *critical* if $\psi'(0+) = 0$ and *subcritical* if $\psi'(0+) > 0$. For a complete overview of the theory of CSBPs the reader is referred to Li [51] and Kyprianou [50, Chapter 12].

Recently, this class of models has been enriched allowing environmental causes to affect the law of reproduction of the population. These processes are known as branching processes in a random environment. The case where the environment is driven by a Lévy process was constructed independently by He et al. [38] and Palau and Pardo [58]. More precisely, let us define the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ as the direct product of the two probability spaces $(\Omega^{(b)}, \mathcal{F}^{(b)}, (\mathcal{F}_t^{(b)})_{t\geqslant 0}, \mathbb{P}^{(b)})$ and $(\Omega^{(e)}, \mathcal{F}^{(e)}, (\mathcal{F}_t^{(e)})_{t\geqslant 0}, \mathbb{P}^{(e)})$ were the branching and environment term are defined, respectively. Therefore, the *continuous-state branching process* $(Z_t, t \geqslant 0)$ *in a Lévy environment* $(S_t, t \geqslant 0)$ is defined as the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} Z_t = \ \ Z_0 \ \ &+ a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s, \end{aligned}$$

where $(B_t^{(b)}, t \geqslant 0)$ is a $(\mathcal{F}_t^{(b)})_{t\geqslant 0}$-adapted standard Brownian motion, $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$

is a $(\mathcal{F}_t^{(b)})_{t\geqslant 0}$-adapted Poisson random measure on $\mathbb{R}_+^3$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$, and $\widetilde{N}^{(b)}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ its compensated version. Further, $(S_t, t \geqslant 0)$ is the Lévy process defined as follows,

$$S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{(-1,1)^c} (e^z - 1)N^{(e)}(\mathrm{d}s,\mathrm{d}z),$$

where $\sigma \geqslant 0$, $\alpha \in \mathbb{R}$, $(B_t^{(e)}, t\geqslant 0)$ is a $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$ - adapted standard Brownian motion, $N^{(e)}(\mathrm{d}s,\mathrm{d}z)$ is a $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$ - Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ with intensity $\mathrm{d}s\pi(\mathrm{d}z)$ and the Lévy measure π satisfies

$$\int_{\mathbb{R}} (1\wedge z^2)\pi(\mathrm{d}z) < \infty.$$

Another Lévy process $(\xi_t, t\geqslant 0)$ appears naturally in this model, which is strongly related with the behaviour of Z. This process is defined as follows

$$\xi_t = \overline{\alpha} t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s,\mathrm{d}z),$$

where

$$\overline{\alpha} := \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z).$$

In particular, under the condition $|\psi'(0+)| < \infty$, it is known that

$$\mathbb{E}_z[Z_t \mid S] = ze^{\xi_t}, \qquad \mathbb{P}_z \text{ -a.s,}$$

see Bansaye et al. [7] . Similarly as in (2), the latter expression allows us to have a classification for this family of processes depending on the behaviour of ξ. To be more precise, we say that the process Z is *subcritical*, *critical* or *supercritical* accordingly as ξ drifts to $-\infty$, oscillates or drifts to $+\infty$. Further, under the condition $|\psi'(0+)| < \infty$, we can compute the Laplace transform of $e^{-\xi_t}Z_t$ (see [58, Proposition 2] or [38, Theorem 3.4]). In other words, if $(v_t(s,\lambda,\xi), s\in[0,t])$ is the unique positive solution of the following backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,\xi) = e^{\xi_s}\psi_0(v_t(s,\lambda,\xi)e^{-\xi_s}), \qquad v_t(t,\lambda,\xi) = \lambda \tag{3}$$

where $\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0+)$. Then for any $\lambda \geqslant 0$ and $t \geqslant s \geqslant 0$, we have

$$\mathbb{E}\Big[\exp\{-\lambda Z_t e^{-\xi_t}\} \,\Big|\, S, \mathcal{F}_s^{(b)}\Big] = \exp\{-Z_s e^{-\xi_s} v_t(s,\lambda,\xi)\}.$$

Moreover, let us denote the random semigroup $h_{s,t}(\lambda) = e^{-\xi_s} v_t(s, \lambda e^{\xi_t}, \xi)$ for all $\lambda \geqslant 0$ and $s \in [0, t]$. Thus,

$$\mathbb{E}\left[e^{-\lambda Z_t} \middle| S, \mathcal{F}_s^{(b)}\right] = \exp\{-Z_s h_{s,t}(\lambda)\},$$

According to [38, Section 2], the mapping $s \mapsto h_{s,t}(\lambda)$ is the unique positive pathwise solution of the integral differential equation

$$h_{s,t}(\lambda) = e^{\xi_t - \xi_s}\lambda - \int_s^t e^{\xi_r - \xi_s} \psi_0\Big(h_{r,t}(\lambda)\Big) \mathrm{d}r, \qquad 0 \leqslant s \leqslant t. \tag{4}$$

For a more complete overview of the theory of CSBPs in a Lévy environment the reader is referred to Chapter 3.

Here we are interested in understanding the asymptotic behaviour of the non-extinction and non-explosion probabilities for this family of processes, under more general conditions than those existing in the literature. The long-term behaviour of the non-extinction probability has been studied, for example, by Li and Xu [52], Palau and Pardo [57] and Palau et al. [59], when the associated branching mechanism corresponds to a stable Lévy process; since the survival probability can be expressed explicitly in terms of the exponential functional of the Lévy process associated with the environment. Recently Bansaye et al. [7] studied the speed of extinction of CSBPs in a critical Lévy environment for more general branching mechanisms. More precisely, the authors in [7] considered the case when the underlying Lévy process in the environmental term satisfies the so-called Spitzer's condition and the branching mechanism is bounded from below by a stable branching mechanism.

In the following two sections we state the main results of this part of the thesis regarding to the explosion problem and the long term behaviour for the probability of non-extinction in CSBPs in a Lévy environment.

Explosion for CSBPs in a Lévy environment

In this section, we state our main results related to the explosion problem for CSBPs in a Lévy environment. Here, we assume that the branching mechanism ψ is given by the negative of the Laplace exponent of a subordinator, i.e. $\psi(\lambda) = -\phi(\lambda)$, where ϕ is a concave, increasing and non-negative function satisfying

$$\phi(\lambda) = \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \mu(\mathrm{d}x), \qquad \lambda \geqslant 0,$$

with
$$\delta := a - \int_{(0,1)} x\mu(\mathrm{d}x) \geqslant 0 \qquad \text{and} \qquad \int_{(0,\infty)} (1 \wedge x)\mu(\mathrm{d}x) < \infty.$$

On the one hand, we study the law of the CSBP in a Lévy environment in the non-finite mean case, i.e. when $\phi'(0+) = \infty$. To be more precise, we deduce the following result.

Theorema 0.0.5. *Assume that $\psi(\lambda) = -\phi(\lambda)$ with $\phi'(0+) = \infty$. For every $z, \lambda, t > 0$ and $x \in \mathbb{R}$, we have*

$$\mathbb{E}_{(z,x)}\Big[\exp\Big\{ - \lambda Z_t e^{-\xi_t}\Big\}\Big|\, \xi\Big] = \exp\Big\{-z v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0)\Big\},$$

where for any $\lambda, t \geqslant 0$, the function $v_t : s \in [0,t] \to v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0)$ is an a.s. solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s, \lambda, \xi) = e^{\xi_s}\psi\Big(v_t(s, \lambda, \xi)e^{-\xi_s}\Big), \qquad a.e. \qquad s \in [0,t]$$

and with terminal condition $v_t(t, \lambda, \xi) = \lambda$.

On the other hand, we provide necessary and sufficient conditions for the process to be conservative, i.e. that the process does not explode in finite time, or in other words that
$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \text{for all} \quad t > 0,$$
where $\mathbb{P}_z$ denotes the law of Z starting in $z > 0$.

Proposition 0.0.6. *Assume that $\psi(\lambda) = -\phi(\lambda)$. A continuous-state branching process in a Lévy environment with branching mechanism ψ is conservative if and only if*
$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \infty.$$

Furthermore, we study the speed of the probability of non-explosion for CSBPs in a Lévy environment, where the associated Lévy process either oscillates or drifts to $-\infty$. First, we present the main result in the case when the environment satisfies Spitzer's condition, that is

$$\frac{1}{t}\int_0^t \mathbb{P}^{(e)}(\xi_s \geqslant 0)\mathrm{d}s \longrightarrow \rho \in (0,1), \qquad \text{as} \qquad t \to \infty,$$

and the branching mechanism satisfies the following condition

$$\text{there exists } \beta \in (-1,0) \text{ and } C < 0 \text{ such that } \ \psi(\lambda) \leqslant C\lambda^{1+\beta} \quad \text{for all } \lambda \geqslant 0. \tag{5}$$

In addition, let us assume

$$\widehat{\mathbb{E}}^{(e)}\Big[H_1 e^{H_1}\Big] < \infty, \tag{6}$$

where $\widehat{\mathbb{E}}^{(e)}$ denotes the expectation associated to the law $\widehat{\mathbb{P}}^{(e)}$ of the dual process $\widehat{\xi} = -\xi$ and H denotes its associated ascending ladder height, we refer to Chapter 3 for further details about the ladder height process.

Theorema 0.0.7 (Critical-explosion regime)**.** *Suppose that ξ satisfies Spitzer's condition with index ρ and that ψ satisfies condition* (5)*. We also assume condition* (6)*. Then, for any $z > 0$, there exists $0 < \mathfrak{C}_1(z) < \infty$ such that*

$$\lim_{t\to\infty} \frac{t^\rho}{\ell(t)} \mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_1(z),$$

where ℓ is a slowly varying function at ∞.

Denote by α_1, π and Φ_ξ, the drift term, the Lévy measure and the Laplace exponent of ξ, respectively. We introduce the following real function

$$A_\xi(x) := -\alpha_1 + \bar{\pi}^{(-)}(-1) + \int_{-x}^{-1} \bar{\pi}^{(-)}(y)\mathrm{d}y, \qquad \text{for} \quad x > 0,$$

where $\bar{\pi}^{(-)}(-x) = \pi(-\infty, -x)$. Recall that μ denotes the measure associated to the branching mechanism. We also introduce the function

$$\widehat{\Phi}_\lambda(u) := \int_{(0,\infty)} \exp\{-\lambda e^u y\}\bar{\mu}(y)\mathrm{d}y,$$

where $\bar{\mu}(y) = \mu(y, \infty)$ Further, let us denote by $\mathtt{E}_1$ the exponential integral, i.e.,

$$\mathtt{E}_1(w) = \int_1^\infty \frac{e^{-wy}}{y}\mathrm{d}y, \qquad w > 0.$$

We can then formulate our second main result regarding to the case when the Lévy process ξ drifts to $-\infty$.

Theorema 0.0.8 (Subcritical-explosion regime)**.** *Suppose that $\Phi_\xi'(0+) < 0$ and*

$$\int_{(a,\infty)} \frac{y}{A_\xi(y)} |\mathrm{d}\widehat{\Phi}_\lambda(y)| < \infty, \qquad \textit{for some} \quad a > 0. \tag{7}$$

Then, for any $z > 0$, there exists $0 < \mathfrak{C}_2(z) < \infty$ such that

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_2(z).$$

Denote by $\widehat{\xi} = -\xi$ *the dual process. In particular, if* $\mathbb{E}^{(e)}\Big[\widehat{\xi}_1\Big] < \infty$, *then the integral condition* (7) *is equivalent to*

$$\int_0^\infty \mathtt{E}_1(\lambda y)\bar{\mu}(y)\mathrm{d}y < \infty.$$

Extinction rates for CSBPs in a subcritical Lévy environment

In this section, we state our main results related to the long-term behaviour of the non-extinction probability for CSBPs in a subcritical Lévy environment. Recall that Φ_ξ denotes Laplace exponent of ξ. As it was observed in [52, 59], there is another phase transition in such regime which depends on whether $\Phi'_\xi(1)$ is less, equal or greater than 0. These regimes are known in the literature as: *strongly*, *intermediate* and *weakly subcritical regime*, respectively.

In order to state our main results we need to introduce the extension of the functional $v_t(s,\lambda,\xi)$ to $s \leq 0$ that appears in He et al. [38, Section 5]. Let us consider an independent copy $(\xi'_t, t \geqslant 0)$ of the Lévy process $(\xi_t, t \geqslant 0)$, thus $\Xi = (\Xi_t, -\infty < t < \infty)$ the time homogeneous Lévy process indexed by $\mathbb{R}$ is defined as follows: $\Xi_0 = \xi_0 = 0$ and

$$\Xi_t = -\lim_{s\downarrow -t}\xi'_s \quad \text{for} \quad t<0 \qquad \text{and} \qquad \Xi_t = \xi_t \quad \text{for} \quad t>0.$$

Next, we use the definition of Ξ to naturally extend the backward differential equation (4) on $s \leqslant 0$. Implicitly, it also follows that for $s \leqslant 0$ the function $s \mapsto h_{s,0}(\lambda)$ is the unique positive pathwise solution to the equation

$$h_{s,0}(\lambda) = e^{-\Xi_s}\lambda - \int_s^0 e^{\Xi_r - \Xi_s}\psi_0\Big(h_{r,0}(\lambda)\Big)\mathrm{d}r, \qquad s \leqslant 0.$$

For the branching mechanism, we require two conditions: the so-called Grey's condition, i.e.

$$\int_1^\infty \frac{1}{\psi_0(\lambda)}\mathrm{d}\lambda < \infty, \tag{8}$$

and the $x\log x$ moment condition for the Lévy measure μ, i.e.

$$\int_1^\infty u\log u\mu(\mathrm{d}u) < \infty. \tag{9}$$

We state now our first main result of this part.

Theorema 0.0.9 (Strongly subcritical regime)**.** *Suppose that conditions* (8) *and* (9) *hold. We assume exponential moments on* ξ *of order 1 and moreover that* $\Phi'_\xi(0) < 0$

and $\Phi'_\xi(1) < 0$. *We also assume*

$$\int_0^\infty \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi'_0\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty.$$

Then for every $z > 0$, *we have*

$$\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z\mathfrak{B}_2,$$

where

$$\mathfrak{B}_2 = \mathbb{E}^{(e,1)}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi'_0(h_{s,0}(\lambda))\mathrm{d}s\right\}\mathrm{d}\lambda\right] \in (0,\infty).$$

Our last main result deals with the intermediate subcritical regime. Here, we assume that the branching mechanism satisfies condition

$$\text{there exists } \beta \in (0,1] \text{ and } C > 0 \text{ such that } \psi_0(\lambda) \geqslant C\lambda^{1+\beta} \quad \text{for } \lambda \geqslant 0. \tag{10}$$

We suppose that the Lévy measure μ satisfies the $x\log x$ moment condition (9). In addition, our arguments require the existence of some exponential moments of the underlying Lévy process ξ, which will be explained and specified in Chapter 5.

Theorema 0.0.10 (Intermediate subcritical regime)**.** *Suppose that conditions* (9) *and* (10) *hold. We assume exponential moments on* ξ *of order strictly bigger than 1 and moreover that* $\Phi'_\xi(0) < 0$ *and* $\Phi'_\xi(1) = 0$. *Finally, we also require that for* $x < 0$

$$\int_0^\infty \mathbb{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-1}^0 \psi'_0(h_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty.$$

Then for every $z > 0$, *we have*

$$\lim_{t\to\infty} t^{3/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z\mathbb{E}^{(e,1)}\Big[H_1\Big]\sqrt{\frac{2}{\pi\Phi''_\xi(1)}}\mathfrak{B}_3,$$

where $(H_t, t \geqslant 0)$ *denotes the ascending ladder process associated to* ξ *and*

$$\mathfrak{B}_3 = \lim_{x\to-\infty} U^{(1)}(-x)\mathbb{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi'_0\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathrm{d}\lambda\right].$$

Part I

Galton-Watson processes in varying environment

Chapter 1

Yaglom's limit for critical GWVE

A Galton–Watson process in varying environment is a discrete time branching process where the offspring distributions vary among generations. Based on a two-spine decomposition technique, we provide a probabilistic argument of a Yaglom-type limit for this family of processes. The result states that, in the critical case, a suitable normalisation of the process conditioned on non-extinction converges in distribution to a standard exponential random variable. The chapter is organised as follows. In Section 1, we recall the definition of Galton-Watson processes in varying environment and present the main theorem of the chapter. In Section 2, we introduce the one-spine and two-spine decompositions. With this in hand, we give an intuitive explanation of the result and we explain why the limit must be exponential. In Section 3, we give some properties of the measures associated with these decompositions and we characterise them via their Laplace transform. Finally, Section 4 contains the proof of Yaglom's Theorem.

1.1 Introduction and main result

A Galton-Watson process in varying environment (GWVE) is a discrete time branching process where the offspring distributions vary among generations, in other words individuals give birth independently and their offspring distributions coincide within each generation. More precisely, a *varying environment* is a sequence $Q = (q_1, q_2, \ldots)$ of probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. A *Galton-Watson process* $Z^Q = \{Z_n^Q : n \geqslant 0\}$ *in a varying environment* Q is a Markov chain defined recursively as follows

$$Z_0^Q = 1 \quad \text{and} \quad Z_n^Q = \sum_{i=1}^{Z_{n-1}^Q} \chi_i^{(n)}, \qquad n \geqslant 1,$$

where $\{\chi_i^{(n)} : i, n \geqslant 1\}$ is a sequence of independent random variables such that

$$\mathbb{P}(\chi_i^{(n)} = k) = q_n(k), \qquad k \in \mathbb{N}_0,\ i,\ n \geqslant 1.$$

The variable $\chi_i^{(n)}$ denotes the offspring of the i-th individual in the $(n-1)$-th generation. Its generating function is given by

$$f_n(s) := \mathbb{E}\left[s^{\chi_i^{(n)}}\right] = \sum_{k=0}^{\infty} s^k q_n(k), \qquad 0 \leqslant s \leqslant 1,\ n \geqslant 1.$$

Hence, by applying the branching property recursively, we deduce that the generating function of Z_n^Q is given in terms of $(f_1, f_2, \ldots)$ as follows

$$\mathbb{E}\left[s^{Z_n^Q}\right] = f_1 \circ \cdots \circ f_n(s), \qquad 0 \leqslant s \leqslant 1,\ n \geqslant 1, \tag{1.1}$$

where $f \circ g$ denotes the composition of f with g.
Moreover, by differentiating in s, we obtain

$$\mathbb{E}[Z_n^Q] = \mu_n, \qquad \text{and} \qquad \frac{\mathbb{E}[Z_n^Q(Z_n^Q - 1)]}{\mathbb{E}[Z_n^Q]^2} = \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}, \qquad n \geqslant 1, \tag{1.2}$$

where $\mu_0 := 1$ and for any $n \geqslant 1$,

$$\mu_n := f_1'(1) \cdots f_n'(1), \qquad \text{and} \qquad \nu_n := \frac{f_n''(1)}{f_n'(1)^2} = \frac{\mathrm{Var}\left[\chi_i^{(n)}\right]}{\mathbb{E}\left[\chi_i^{(n)}\right]^2} + \left(1 - \frac{1}{\mathbb{E}\left[\chi_i^{(n)}\right]}\right), \tag{1.3}$$

where $\mathrm{Var}\left[\chi_i^{(n)}\right]$ is the variance of the variable. For further details about GWVEs, we refer to the monograph of Kersting and Vatutin [45].

According with Kersting, [44], we say that a GWVE is *regular* if there exists a constant $c > 0$ such that for all $n \geqslant 1$,

$$\mathbb{E}\left[(\chi_i^{(n)})^2 \mathbf{1}_{\{\chi_i^{(n)} \geqslant 2\}}\right] \leqslant c\mathbb{E}\left[\chi_i^{(n)} \mathbf{1}_{\{\chi_i^{(n)} \geqslant 2\}}\right] \mathbb{E}\left[\chi_i^{(n)} \middle| \chi_i^{(n)} \geqslant 1\right].$$

He proved that a regular GWVE has extinction a.s, namely $\eta := \mathbb{P}(Z_n^Q = 0 \text{ for some } n) = 1$, if and only if $\sum_{k=0}^{\infty} \frac{\nu_{k+1}}{\mu_k} = \infty$ or $\mu_n \to 0$ as $n \to \infty$, [44, Theorem 1]. In addition, he gave the following classification. If $\eta < 1$ and $\lim\limits_{n\to\infty} \mathbb{E}[Z_n^Q] = \infty$, we say that Z^Q is *supercritical*. Further, if $\eta < 1$ and $\lim\limits_{n\to\infty} \mathbb{E}[Z_n^Q] < \infty$, the process Z^Q is called *asymptotically degenerate*. On other hand, when $\eta = 1$ we say that Z^Q is either *critical* or *subcritical* if either $\lim\limits_{n\to\infty} \mathbb{E}[Z_n^Q | Z_n^Q > 0] = \infty$ or $\lim\limits_{n\to\infty} \mathbb{E}[Z_n^Q | Z_n^Q > 0] < \infty$, respectively.

Furthermore, these regimes are characterised as follow (see [44, Proposition 1]):

A regular GWVE is

i. *supercritical* if and only if $\sum\limits_{k=0}^{\infty} \frac{\nu_{k+1}}{\mu_k} < \infty$ and $\lim\limits_{n\to\infty} \mu_n = \infty$,

ii. *asymptotically degenerate* if and only if $\sum\limits_{k=0}^{\infty} \frac{\nu_{k+1}}{\mu_k} < \infty$ and $0 < \lim\limits_{n\to\infty} \mu_n < \infty$,

iii. *critical* if and only if $\sum\limits_{k=0}^{\infty} \frac{\nu_{k+1}}{\mu_k} = \infty$ and $\lim\limits_{n\to\infty} \mu_n \sum\limits_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} = \infty$,

iv. *subcritical* if and only if $\liminf\limits_{n\to\infty} \mu_n = 0$ and $\liminf\limits_{n\to\infty} \mu_n \sum\limits_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} < \infty$.

Kersting's definition is an extension of the classical categorisation of branching processes. Indeed, when the environment is constant, we have $\mu_k = \mu^k$ and $\nu_k = \sigma^2/\mu^2 + (1 - 1/\mu)$, for $k \geqslant 1$, where μ and σ^2 are the mean and variance of the offspring distribution, respectively; we recover the original classification. We observe that in this case, the asymptotically degenerate case is not possible.

Given a varying environment Q, we define the sequence $\{a_n^Q : n \geqslant 0\}$ as follows

$$a_0^Q = 1, \qquad \text{and} \qquad a_n^Q = \frac{\mu_n}{2} \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}, \qquad n \geqslant 1.$$

Kersting, [44, Theorem 4], showed that in the critical regime, $a_n^Q \to \infty$ and that

$$\lim_{n\to\infty} \frac{a_n^Q}{\mu_n} \mathbb{P}(Z_n^Q > 0) = 1. \tag{1.4}$$

This asymptotic behaviour is a generalisation of Kolmogorov's theorem for Galton-Watson processes with constant environment (see [47]).

In the rest of the chapter, we work with regular critical GWVE. Further, we assume the following condition

$$\text{there exists } c > 0 \text{ such that } \ f_n'''(1) \leqslant c f_n''(1)(1 + f_n'(1)), \ \text{ for any } n \geqslant 1. \tag{$\mathbf{A^*}$}$$

Kersting proved that this condition implies that the GWVE is regular, see [44, Proposition 2]. Moreover, he explained that Condition ($\mathbf{A^*}$) is a rather mild condition. Indeed, it is satisfied by most common probability distributions, for instance the Poisson, binomial, geometric, hypergeometric, and negative binomial distributions.

Another important example satisfying Condition (**A***) are random variables that are a.s. uniformly bounded by a constant.

We are ready to present our main result, which generalises Yaglom's theorem for classical Galton-Watson processes.

Theorema 1.1.1 (Yaglom's limit). *Let* $\{Z_n^Q : n \geqslant 0\}$ *be a critical GWVE that satisfies Condition* (**A***). *Then*

$$\left(\frac{Z_n^Q}{a_n^Q}; \mathbb{P}(\,\cdot\,|Z_n^Q > 0)\right) \xrightarrow{(d)} (Y; \mathbb{P}), \qquad \text{as } n \to \infty,$$

where Y *is a standard exponential random variable.*

In the classical theory with constant environment, this result has several proofs, the first one was given by Yaglom [75]. In [55], a probabilistic proof via a characterisation of the exponential distribution was presented. Later on, Geiger characterised the exponential random variable by a distributional equation and he presented another proof of Yaglom's limit based on that equation (see [31, 32]). Recently, Ren et al. [67], developed yet another new proof using a two-spine decomposition technique.

When the environment is varying, Jagers [42] proved the convergence under extra assumptions. Afterwards, Bhattacharya and Perlman [14] obtained the same result with weaker assumptions than Jagers (but stronger than ours). Kersting [44] provided yet another proof in a similar framework to ours, that we will explain below. An extension in the presence of immigration and the same setting as Kersting's has been established in [34]. A multi-type version with analogous assumptions as Kersting's can be found in [25]. All these authors established the exponential convergence using an analytical approach. The condition in Kersting [44] is the following. For every $\epsilon > 0$ there is a constant $c_\epsilon < \infty$ such that

$$\mathbb{E}\left[(\chi_i^{(n)})^2 \mathbf{1}_{\left\{\chi_i^{(n)} > c_\epsilon(1+\mathbb{E}[\chi_i^{(n)}])\right\}}\right] \leqslant \epsilon \mathbb{E}\left[(\chi_i^{(n)})^2 \mathbf{1}_{\{\chi_i^{(n)} \geqslant 2\}}\right], \qquad \text{for any } n \geqslant 1.$$

He explained that a direct verification of this assumptions can be cumbersome. Therefore, he introduced Condition (**A***) which is easier to handle and implies the latter condition. For this reason, we prefer to work directly under Assumption (**A***), which is good enough for our purposes.

In this manuscript, we give a probabilistic argument of Yaglom's limit for GWVE. It is based on a two spine decomposition method and a characterisation of the exponential distribution via a size-biased transform and is close in spirit to that of [67].

A one-spine decomposition is already known in the literature (see [45, Section 1.4]). We believe that it is possible to use a one-spine decomposition to prove Yaglom-type limit for GWVE, but we could not find a proof with this approach in the literature. However, we decided to tackle the proof with a two spines decomposition. The reason comes from the classical theory of Galton-Watson processes in constant environment. Consider the most recent common ancestor (MRCA) of the particles at generation n. When the environment is constant, Geiger [32] showed that conditioned on the event of non-extinction at generation n, asymptotically there are exactly two children of the MRCA with at least one descendant at generation n. Based on this intuition, it is natural to consider a two spine decomposition whose spines correspond to the genealogical lines of these two individuals.

The authors in [67] created a two-spine decomposition technique for Galton-Watson processes in constant environment that cannot be applied directly into our settings. Here, associated to each Z_n^Q, we construct a Galton-Watson tree in varying environment up to time n with two marked genealogical lines. This tree can be decomposed in subtrees along these lines. A key point is the distribution of the generation of the most recent common ancestor of these genealogical lines, here denoted by K_n. When the environment is constant, K_n has uniform distribution on $\{0, \ldots, n-1\}$ and the subtrees are independent Galton-Watson trees. When the environment varies, this last property does not hold anymore. In order to match the above decomposition with that at the exponential distribution, it is fundamental to know the law of K_n explicitly. Thus, we determine the distribution of K_n that makes the method work. Moreover, we identify the subtrees with Galton-Watson trees in a modified environment. In the next section, we explain this in further detail.

Our contribution is that our proof provides further understanding on why the limit must be an exponential random variable. An important part of our approach is in studying random trees and being able to adequately select inside them two marked genealogical lines. We believe that one can adapt this decomposition technique to establish a Yaglom-type limit for branching processes in random environment, i.e., when the environment is given by a sequence of random probability measures on $\mathbb{N}_0$. If the random environment is an i.i.d. sequence of probability measures, the Yaglom-type limit theorem under a quenched approach is known in the literature [45, Theorem 6.2]. In particular, they showed that when the environment is given by linear fractional distributions, the Yaglom-type limit is an exponential random variable. Then, for these and other distributions the construction has to be the same but, for the two genealogical lines, one has to find the distribution of the generation of their most recent common ancestor that makes the method work. Furthermore, by using the approach of several spines decomposition it would be possible to study the genealogy of Galton-Watson processes in varying environment. For the moment this technique has only been done in the constant environment case (see [36]). These possible applications highlights the potential and relevance of our methodology.

1.2 Outline of the proof

In this section, we provide an intuitive explanation of the result and explain why the limit must be an exponential random variable. First, we explain the one-spine and two-spines decompositions. Then, we relate them with a size-biased characterisation of the exponential random variable.

Recall that given a random variable X and a Borel function g such that $\mathbb{P}(g(X) \geqslant 0) = 1$, and $\mathbb{E}[g(X)] \in (0,\infty)$, we say that W is a $g(X)$-*transform* of X if

$$\mathbb{E}[f(W)] = \frac{\mathbb{E}[f(X)g(X)]}{\mathbb{E}[g(X)]},$$

for each positive Borel function f. If $g(x) = x$, we also call it the *size-biased transform.*

Observe that the law of a non-negative random variable X conditioned on being strictly positive can be described in terms of its size-biased transform. More precisely, for each $\lambda \geqslant 0$,

$$\mathbb{E}\left[1 - e^{-\lambda X} \mid X > 0\right] = \int_0^\lambda \frac{\mathbb{E}\left[Xe^{-sX}\right]}{\mathbb{P}(X > 0)} \mathrm{d}s = \mathbb{E}\left[X \mid X > 0\right] \int_0^\lambda \mathbb{E}\left[e^{-s\dot{X}}\right] \mathrm{d}s, \qquad (1.5)$$

where $\dot{X}$ is the size-biased transform of X. Recall that a sequence of non-negative random variables converges in distribution if and only if their Laplace transforms converge. As a consequence, we obtain the following lemma

Lemma 1.2.1. *Let $\{X_n : n \geqslant 0\}$ be a sequence of non-negative random variables. Then the variables conditioned on being strictly positive $\{X_n\,; \mathbb{P}(\cdot \mid X_n > 0)\}_{n\geqslant 0}$ converge in distribution to a strictly positive random variable Y if and only if $\mathbb{E}\left[X_n \mid X_n > 0\right] \to \mathbb{E}\left[Y\right]$ and $\dot{X}_n$ converges in distribution to $\dot{Y}$, where $\dot{X}_n$ and $\dot{Y}$ are the size-biased transforms of X_n and Y, respectively.*

By Lemma 1.2.1, in order to prove Theorem 1.1.1 we need to study the size-biased process $\dot{Z}^Q := \{\dot{Z}^Q_n : n \geqslant 0\}$. Recall that there is a relationship between Galton-

Watson processes in environment Q and Galton-Watson trees in environment Q. In the tree, any particle or individual in generation i gives birth to particles in generation $i+1$ according to q_{i+1}. The variable Z_n^Q is the number of particles at generation n in the tree. In a similar way, $\dot{Z}_n^Q$ is the population size at generation n of some random tree. According to Kersting and Vatutin [45, Sections 1.4.1 and 1.4.2], the tree associated to $\dot{Z}^Q$ is a size-biased tree in varying environment Q. More precisely, for each $i \geqslant 1$, let $\dot{q}_i$ be the size-biased transform of q_i,

$$\dot{q}_i(k) = \frac{k}{f_i'(1)} q_i(k), \qquad k \in \mathbb{N}_0. \tag{1.6}$$

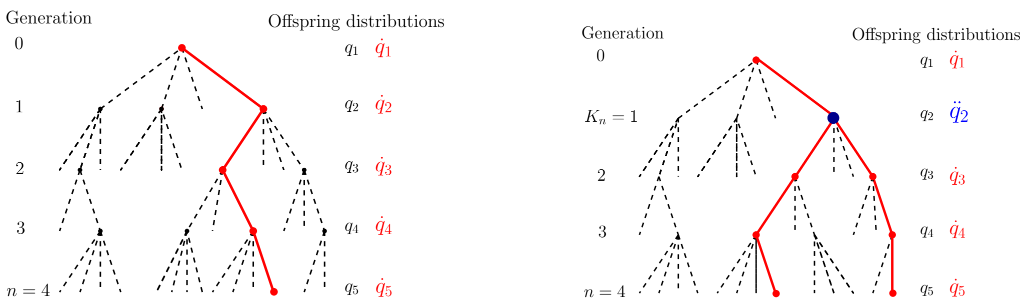
The *size-biased tree in environment* Q is constructed as follows:

(i) We first establish an initial marked particle,

(ii) the marked particle in generation $i \in \mathbb{N}_0$ gives birth to particles in generation $i+1$ according to $\dot{q}_{i+1}$. Uniformly, we select one of these particles as the marked particle. All the others particles are unmarked,

(iii) any unmarked particle in generation $i \in \mathbb{N}_0$ gives birth to unmarked particles in generation $i+1$ according to q_{i+1}, independently of other particles.

The marked genealogical line is called *spine*. This construction is known as the *one-spine decomposition*; see Figure 1.1a below. The constant environment case was done by Lyons, Pemantle and Peres [55]. According to Kersting and Vatutin, $\dot{Z}_n^Q$ is the number of particles at generation n in this tree.



(a) One-spine decomposition (b) Two-spine decomposition

Fig. 1.1 Spine decompositions

Now, we want to construct a random tree up to generation n with two marked genealogical lines or spines. Denote by K_n the *generation of the most recent common* *ancestor of the lines*. Note that before K_n there is only one spine and in generation $K_n + 1$ a second spine is created. Since the offspring distribution is varying among generations, K_n should depend on the environment. We assume that in this construction, K_n has the following distribution

$$\mathbb{P}(K_n = r) := \frac{\nu_{r+1}}{\mu_r}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1}, \qquad 0 \leqslant r \leqslant n-1, \tag{1.7}$$

where μ_n and ν_n are defined in (1.3). Thus, by (1.3), generations with larger offspring mean or larger offspring variance are more probably to be chosen as K_n. In generation K_n, we need to have an offspring distribution with two or more individuals. We denote by $\ddot{q}_i$ the $q_i(q_i-1)$-transform of q_i given by

$$\ddot{q}_i(k) = \frac{k(k-1)q_i(k)}{\nu_i f_i'(1)^2}, \qquad k \in \mathbb{N}_0, \quad i = 1, \ldots, n. \tag{1.8}$$

We define a *$X(X-1)$-type size-biased tree in environment Q up to time n* as the tree constructed as follows:

(i) we first establish an initial marked particle,

(ii) select K_n according to (1.7),

(iii) the marked particle in generation K_n gives birth to particles according to $\ddot{q}_{K_n+1}$. Uniformly without replacement, we select two of these particles as the marked particles in generation $K_n + 1$. The other particles are unmarked,

(iv) any marked particle in generation $i \in \{0, \ldots, n-1\} \setminus K_n$ gives birth to particles in generation $i+1$ according to $\dot{q}_{i+1}$. Uniformly, select one of these as the marked particle. All the other particles are not marked,

(v) any unmarked particle in generation $i \in \{0, \ldots, n-1\}$ gives birth to unmarked particles in generation $i+1$ according to q_{i+1}, independently of other particles.

We call this construction as the *two-spine decomposition*; see Figure 1.1b. Ren et. al [67] provided a two spine decomposition for Galton-Watson processes in a constant environment. In this case, the distribution of K_n is uniform on $\{0, \ldots, n-1\}$. Using that the environment is constant we can recover their construction.

With these constructions, we can give an intuitive explanation of why the limit must be an exponential random variable, we will make this intuition rigorous in the following sections. For any $0 \leqslant k \leqslant n$, let $\ddot{Z}_k^Q$ be the population size at the k-th generation in the previous tree. From the constructions of the size-biased trees (see Figure 1.1), we see that we can decompose the particles associated to $\ddot{Z}_n^Q$ into descendants attached to the longer spine and descendants attached to the shorter spine. The descendants attached to the longer spine are approximately distributed as the population in the n-th generation of a size-biased tree with environment Q, while the descendants of the shorter spine are approximately distributed as the population in generation $n-(K_n+1)$ of a size-biased tree with environment $Q_{K_n+1} := (q_{K_n+2}, q_{K_n+3}, \ldots)$. By construction, the two subpopulations are independent. Therefore, we have roughly that

$$\ddot{Z}_n^Q \overset{(d)}{\approx} \dot{Z}_n^Q + \dot{Z}_{n-(K_n+1)}^{Q_{K_n+1}}, \qquad n \geqslant 1, \tag{1.9}$$

where the right-hand side of the equation is an independent sum. If we normalise with a_n^Q, we obtain

$$\frac{\ddot{Z}_n^Q}{a_n^Q} \overset{(d)}{\approx} \frac{\dot{Z}_n^Q}{a_n^Q} + \frac{a_{n-(K_n+1)}^{Q_{K_n+1}}}{a_n^Q} \frac{\dot{Z}_{n-(K_n+1)}^{Q_{K_n+1}}}{a_{n-(K_n+1)}^{Q_{K_n+1}}}, \qquad n \geqslant 1. \tag{1.10}$$

Kersting and Vatutin [45, Lemma 1.2] proved that $\dot{Z}_n^Q$ is the size-biased transform of Z_n^Q. In this work, we provide a precise meaning of equation (1.9), we prove that $\ddot{Z}_n^Q$ is the $Z_n^Q(Z_n^Q-1)$-transform of Z_n^Q (see Proposition 1.3.1), that

$$(a_n^Q)^{-1} a_{n-(K_n+1)}^{Q_{K_n+1}} \xrightarrow{(d)} U, \qquad \text{as } n \to \infty,$$

where U is a uniform random variable on $[0,1]$ (see Proposition 1.4.1), and that $\dot{Z}_n^Q/a_n^Q$ converges in distribution to a random variable $\dot{Y}$ (see Proposition 1.4.2).

Since $\ddot{Z}_n^Q$ is the $(\dot{Z}_n^Q-1)$-transform of $\dot{Z}_n^Q$, we have that $\ddot{Z}_n^Q/a_n^Q$ converges in distribution to $\ddot{Y}$, the $\dot{Y}$-transform of $\dot{Y}$. Hence, by Lemma 1.2.1, if we take limits in (1.10), we see that Z_n^Q/a_n^Q conditioned on being strictly positive converges in distribution to a random variable Y that satisfies

$$\ddot{Y} \overset{(d)}{=} \dot{Y} + U \cdot \dot{Y}' \tag{1.11}$$

where $\dot{Y}$ and $\dot{Y}'$ are both Y-transforms of Y, $\ddot{Y}$ is a Y^2-transform of Y, and U is a uniform random variable on $[0,1]$ independent of $\dot{Y}$ and $\dot{Y}'$. Ren et. al. [67, Lemma 1.3], showed that a variable Y is exponentially distributed with mean 1 if and only if (1.11) holds. Therefore, Z_n^Q/a_n^Q must converge in distribution to a standard exponential random variable.

1.3 Size-biased trees

In this section, we study the size-biased trees defined in the previous section. We associate them to probability measures in the set of rooted trees. For this purpose, we introduce the so-called *Ulam-Harris labeling*. Let $\mathcal{U}$ be the set of finite sequences of strictly positive integers, including $\emptyset$. For $u \in \mathcal{U}$, we define the length of u by $|u| := n$, if $u = u_1 \cdots u_n$, where $n \geqslant 1$ and by $|\emptyset| := 0$ if $u = \emptyset$. If u and v are two elements in $\mathcal{U}$, we denote by uv the concatenation of u and v, with the convention that $uv = u$ if $v = \emptyset$. The genealogical line of u is denoted by $[\emptyset, u] = \{\emptyset\} \cup \{u_1 \cdots u_j : j = 1, \ldots, n\}$. Let $\mathbf{s} \subset \mathcal{U}$, its most recent common ancestor is the unique element $v \in \cap_{u \in \mathbf{s}} [\emptyset, u]$ with maximal length and its generation is denoted by $K_{\mathbf{s}}$.

A *rooted tree* $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies $\emptyset \in \mathbf{t}$, $[\emptyset, u] \subset \mathbf{t}$ for any $u \in \mathbf{t}$, and if $u \in \mathbf{t}$ and $i \in \mathbb{N}$ satisfy that $ui \in \mathbf{t}$ then, $uj \in \mathbf{t}$ for all $1 \leqslant j \leqslant i$. Denote by $\mathcal{T} = \{\mathbf{t} : \mathbf{t} \text{ is a tree}\}$, the subspace of rooted trees. The vertex $\emptyset$ is called *the root* of the tree. For any $u \in \mathbf{t}$, we define the number of offspring of u by $l_u(\mathbf{t}) = \max\{i \in \mathbb{Z}^+ : ui \in \mathbf{t}\}$. The height of $\mathbf{t}$ is defined by $|\mathbf{t}| = \sup\{|u| : u \in \mathbf{t}\}$. For any $n \in \mathbb{N}$ and $\mathbf{t}, \tilde{\mathbf{t}}$ trees, we write $\mathbf{t} \overset{n}{=} \tilde{\mathbf{t}}$ if they coincide up to height n. The population size in the n-th generation of the tree $\mathbf{t}$ is denoted by $X_n(\mathbf{t}) = \#\{u \in \mathbf{t} : |u| = n\}$.

A *Galton-Watson tree in the environment* $Q = (q_1, q_2, \ldots)$ is a $\mathcal{T}$-valued random variable $\mathbf{T}$ such that

$$\mathbf{G}_n(\mathbf{t}) := \mathbb{P}(\mathbf{T} \overset{n}{=} \mathbf{t}) = \prod_{u \in \mathbf{t} \colon |u| < n} q_{|u|+1}(l_u(\mathbf{t})),$$

for any $n \geqslant 0$ and any tree $\mathbf{t}$. As we said before, the process $Z = \{Z_n^Q : n \geqslant 0\}$ defined as $Z_n^Q = X_n(\mathbf{T})$ is a Galton-Watson process in environment Q.

Now, we deal with the one-spine decomposition. This construction builds a tree along a distinguished path. More precisely, a *spine* or distinguished path $\mathbf{v}$ on a tree $\mathbf{t}$ is a sequence $\{v^{(k)} : k = 0, 1, \ldots, |\mathbf{t}|\} \subset \mathbf{t}$ (or $\{v^{(k)} : k = 0, 1, \ldots\} \subset \mathbf{t}$ if $|\mathbf{t}| = \infty$) such that $v^{(0)} = \emptyset$ and $v^{(k)} = v^{(k-1)} j$ for some $j \in \mathbb{N}$, for any $1 \leqslant k \leqslant |\mathbf{t}|$. We denote by $\dot{\mathcal{T}}$, the *subspace of trees with one spine*

$$\dot{\mathcal{T}} = \{(\mathbf{t}, \mathbf{v}) : \ \mathbf{t} \text{ is a tree and } \mathbf{v} \text{ is a spine on } \mathbf{t}\}$$

and by $\mathcal{T}_n = \{\mathbf{t} \in \mathcal{T} : |\mathbf{t}| = n\}$ and $\dot{\mathcal{T}}_n = \{(\mathbf{t}, \mathbf{v}) \in \dot{\mathcal{T}} : |\mathbf{t}| = n\}$ the restriction of $\mathcal{T}$ and $\dot{\mathcal{T}}$ to trees with height n.

We are going to construct the probability distribution of the size-biased tree in the environment Q on the state space $\mathcal{T}$. First, we need to define a probability distribution on $\dot{\mathcal{T}}$. Recall the construction of the size-biased tree in the previous section; individuals along the spine, $\{u \in \mathbf{t} : u \in \mathbf{v}\}$, have offspring distribution $\dot{q}_{|u|+1}$ given by (1.6), and from their offspring we select one uniformly as the spine individual in the next generation. Individuals outside the spine, $\{u \in \mathbf{t} : u \notin \mathbf{v}\}$, have offspring distribution $q_{|u|+1}$. Then, the size-biased tree can be seen as a $\dot{\mathcal{T}}$-valued random variable $(\dot{\mathbf{T}}, \mathbf{V})$ with distribution

$$\mathbb{P}((\dot{\mathbf{T}}, \mathbf{V}) \overset{n}{=} (\mathbf{t}, \mathbf{v})) := \prod_{u \in \mathbf{v} \colon |u|<n} \dot{q}_{|u|+1}(l_u(\mathbf{t})) \frac{1}{l_u(\mathbf{t})} \prod_{u \in \mathbf{t} \setminus \mathbf{v} \colon |u|<n} q_{|u|+1}(l_u(\mathbf{t})),$$

for any $n \geqslant 0$ and any $(\mathbf{t}, \mathbf{v}) \in \dot{\mathcal{T}}_n$. One readily checks that this measure is a probability on $\dot{\mathcal{T}}$ by using the definition of $\dot{q}$ and the fact that $\mathbf{G}_n$ is a probability measure. In a similar way, we can write

$$\mathbb{P}((\dot{\mathbf{T}}, \mathbf{V}) \overset{n}{=} (\mathbf{t}, \mathbf{v})) = \frac{1}{\mu_n} \cdot \mathbf{G}_n(\mathbf{t}), \quad (\mathbf{t}, \mathbf{v}) \in \dot{\mathcal{T}}.$$

Hence, by summing over all the possible spines, we obtain the distribution of the *size-biased Galton-Watson tree in environment* Q on $\mathcal{T}$

$$\dot{\mathbf{G}}_n(\mathbf{t}) := \mathbb{P}(\dot{\mathbf{T}} \overset{n}{=} \mathbf{t}) = \sum_{\mathbf{v}:(\mathbf{t},\mathbf{v}) \in \dot{\mathcal{T}}_n} \mathbb{P}((\dot{\mathbf{T}}, \mathbf{V}) \overset{n}{=} (\mathbf{t}, \mathbf{v})) = \frac{1}{\mu_n} X_n(\mathbf{t}) \cdot \mathbf{G}_n(\mathbf{t}),$$

for any $n \geqslant 0$ and any $\mathbf{t} \in \mathcal{T}_n$ (see also [45, Lemma 1.2]). Define the process $\dot{Z}^Q = \{\dot{Z}_n^Q : n \geqslant 0\}$ as $\dot{Z}_n^Q = X_n(\dot{\mathbf{T}})$, for each $n \geqslant 1$. Then, by using the measure $\dot{\mathbf{G}}_n$ we can see that the process $\{\dot{Z}_m^Q : 0 \leqslant m \leqslant n\}$ is a Z_n^Q-transform of $\{Z_m^Q : 0 \leqslant m \leqslant n\}$, in other words

$$\mathbb{E}\left[g(\dot{Z}_1^Q, \ldots, \dot{Z}_n^Q)\right] = \frac{\mathbb{E}\left[Z_n^Q g(Z_1^Q, \ldots, Z_n^Q)\right]}{\mathbb{E}\left[Z_n^Q\right]}, \qquad \text{for all bounded functions } g.$$

Now we consider the probability distribution associated to the $X(X-1)$-type size-biased tree up to time n on the state space $\mathcal{T}_n$. As we did before, we define a measure on

$$\ddot{\mathcal{T}}_n := \left\{(\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}}) : \ (\mathbf{t}, \mathbf{v}), (\mathbf{t}, \tilde{\mathbf{v}}) \in \dot{\mathcal{T}}_n, \ \mathbf{v} \neq \tilde{\mathbf{v}}\right\}, \qquad n \in \mathbb{N},$$

the *subspace of trees with height* n *and two different spines.* Given a $(\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}}) \in \ddot{\mathcal{T}}_n$, we denote by $K_{\mathbf{v},\tilde{\mathbf{v}}} = \max\{r < n : \mathbf{v} \overset{r}{=} \tilde{\mathbf{v}}\}$ the generation of the most recent common ancestor of $\mathbf{v} \cup \tilde{\mathbf{v}}$.

Recall the construction of a $X(X-1)$-type size-biased tree in the previous section; (i) consider an initial spine individual, (ii) select the generation of the most recent common ancestor, $K_{\mathbf{v},\tilde{\mathbf{v}}}$, according to (1.7), (iii) the spine individual u in that generation has offspring distribution $\ddot{q}_{|u|+1}$ given by (1.8). From its offspring we select uniformly without replacement two as spine individuals in the next generation, (iv) the spine individuals in the other generations, $\{u \in \mathbf{v} \cup \tilde{\mathbf{v}} : |u| \neq K_{\mathbf{v},\tilde{\mathbf{v}}}\}$, have offspring distribution $\dot{q}_{|u|+1}$ given by (1.6). From its offspring we select uniformly one as the spine individual in the next generation, (v) finally, individuals outside the spine, $\{u \in \mathbf{t} : u \notin \mathbf{v} \cup \tilde{\mathbf{v}}\}$, have offspring distribution $q_{|u|+1}$. Then, the $X(X-1)$-type size-biased tree up to time n can be seen as a $\ddot{\mathcal{T}}_n$-valued random variable $(\ddot{\mathbf{T}}, \mathbf{V}, \widetilde{\mathbf{V}})$ with distribution

$$\begin{aligned}
&\mathbb{P}((\ddot{\mathbf{T}}, \mathbf{V}, \widetilde{\mathbf{V}}) \overset{n}{=} (\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}})) \\
&\qquad := \frac{\nu_{K_{\mathbf{v},\tilde{\mathbf{v}}}+1}}{\mu_{K_{\mathbf{v},\tilde{\mathbf{v}}}}} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} \right)^{-1} \prod_{u \in \mathbf{v} \cup \tilde{\mathbf{v}} :\, K_{\mathbf{v},\tilde{\mathbf{v}}} = |u|} \ddot{q}_{|u|+1}(l_u(\mathbf{t})) \frac{2}{l_u(\mathbf{t})(l_u(\mathbf{t})-1)} \\
&\qquad\qquad \prod_{u \in \mathbf{v} \cup \tilde{\mathbf{v}} :\, K_{\mathbf{v},\tilde{\mathbf{v}}} \neq |u| < n} \dot{q}_{|u|+1}(l_u(\mathbf{t})) \frac{1}{l_u(\mathbf{t})} \prod_{u \in \mathbf{t} \setminus (\mathbf{v} \cup \tilde{\mathbf{v}}) :\, |u| < n} q_{|u|+1}(l_u(\mathbf{t})),
\end{aligned}$$

for any $(\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}}) \in \ddot{\mathcal{T}}_n$. Here, the first two terms in the right-hand side of the equation are associated with step (ii). The first product is associated with step (iii). Then, in the second line, the first product is obtained with (iv). Finally, we use (v) to obtain the last product. By using the definition of q, $\dot{q}$ and $\ddot{q}$, one can readily verify that the previous expression defines a probability measure on $\ddot{\mathcal{T}}_n$. Moreover, we have

$$\mathbb{P}((\ddot{\mathbf{T}}, \mathbf{V}, \widetilde{\mathbf{V}}) \overset{n}{=} (\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}})) = \frac{2}{\mu_n^2} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} \right)^{-1} \mathbf{G}_n(\mathbf{t}),$$

for any $(\mathbf{t}, \mathbf{v}, \tilde{\mathbf{v}}) \in \ddot{\mathcal{T}}_n$. Then, by summing over all the possible two spines, we obtain that the $X(X-1)$-type size-biased tree up to time n is a $\mathcal{T}_n$-valued random variable $\ddot{\mathbf{T}}$ with law

$$\ddot{\mathbf{G}}_n(\mathbf{t}) := \mathbb{P}(\ddot{\mathbf{T}} \overset{n}{=} \mathbf{t}) = \frac{1}{\mu_n^2} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} \right)^{-1} X_n(\mathbf{t})(X_n(\mathbf{t})-1) \cdot \mathbf{G}_n(\mathbf{t}), \tag{1.12}$$

for any $\mathbf{t} \in \mathcal{T}_n$. Define the process $\ddot{Z}^Q = \{\ddot{Z}_m^Q : 0 \leqslant m \leqslant n\}$ by $\ddot{Z}_m^Q = X_m(\ddot{\mathbf{T}})$.

Opposite to what happens with $(\dot{\mathbf{G}}_n : n \geqslant 1)$, by construction, the measures $(\ddot{\mathbf{G}}_n : n \geqslant 1)$ are not consistent in the sense that $\ddot{\mathbf{G}}_n$ is not a restriction of $\ddot{\mathbf{G}}_{n+1}$ to the tree with size n. More precisely, in the size-biased tree, the change of measure is intuitively a martingale since the tree under this measure has one spine throughout all generations. While, in the $X(X-1)$-type size-biased tree, if we restrict a tree with two spines at time n to the previous generations it is possible to lose one spine. Indeed, the tree will have only one marked particle in all the generations before $K_{\mathbf{v},\tilde{\mathbf{v}}}$. Then, the change of measure in the next proposition is not a martingale change of measure, not even in the case of constant environment [67, Theorem 1.2]. However, it allows us to conclude that $\{\ddot{Z}^Q_m : 0 \leqslant m \leqslant n\}$ is a $Z^Q_n(Z^Q_n - 1)$-transform of $\{Z^Q_m : 0 \leqslant m \leqslant n\}$.

Proposition 1.3.1. *Let* $\{Z^Q_n : n \geqslant 0\}$ *be a GWVE and for any* $n \in \mathbb{N}_0$, *let* $\ddot{Z}^Q = (\ddot{Z}^Q_m : 0 \leqslant m \leqslant n)$ *be the process associated with the* $X(X-1)$*-type size-biased tree up to time* n. *Then, for any bounded function* $g : \mathbb{Z}^n_+ \to \mathbb{R}$,

$$\mathbb{E}[g(\ddot{Z}^Q_1, \dots, \ddot{Z}^Q_n)] = \frac{\mathbb{E}[Z^Q_n(Z^Q_n - 1)g(Z^Q_1, \dots, Z^Q_n)]}{\mathbb{E}[Z^Q_n(Z^Q_n - 1)]}. \tag{1.13}$$

Proof. Fix $n \geqslant 0$ and recall that for each $m \leqslant n$, $Z^Q_m = X_m(\mathbf{T})$ under the measure $\mathbf{G}_n$ and $\ddot{Z}^Q_m = X_m(\mathbf{T})$ under the measure $\ddot{\mathbf{G}}_n$. Hence, by (1.12)

$$\begin{aligned}
\mathbb{E}[g(\ddot{Z}^Q_1, \dots, \ddot{Z}^Q_n)] &= \ddot{\mathbf{G}}_n[g(X_1(\mathbf{T}), \dots, X_n(\mathbf{T}))] \\
&= \frac{1}{\mu_n^2}\left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} \mathbf{G}_n\left[X_n(\mathbf{T})(X_n(\mathbf{T}) - 1)g(X_1(\mathbf{T}), \dots, X_n(\mathbf{T}))\right] \\
&= \frac{1}{\mu_n^2}\left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} \mathbb{E}\left[Z^Q_n(Z^Q_n - 1)g(Z^Q_1, \dots, Z^Q_n)\right].
\end{aligned}$$

By taking $g \equiv 1$, we deduce that

$$\mathbb{E}[Z^Q_n(Z^Q_n - 1)] = \mu_n^2 \sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k},$$

which implies the result. □

In the reminder of this section, we study some properties of the previous decompositions. We first introduce the notation to refer to shifted environments. Let q be a probability measure on $\mathbb{N}_0$ such that $q(\{0, 1, \dots, r-1\}) = 0$ for some $r \in \mathbb{N}$. We define the probability measure $[q - r]$ in $\mathbb{N}_0$ by $[q - r](i) = q(i + r)$ for all $i \in \mathbb{N}_0$.

Given a probability measure q and an environment $Q=(q_1,q_2,...)$, we denote

$$q\oplus Q:=(q,q_1,q_2,\ldots).$$

For any $m\in\mathbb{N}_0$, as in Section 1.2 we set

$$Q_m:=(q_{m+1},q_{m+2},...).$$

We can compute the Laplace transform of $\ddot{Z}_n^Q$ in terms of the Laplace transform of $\dot{Z}_n^Q$ and $\dot{Z}_{n-(m+1)}^{Q_{m+1}}$, as indicated below. The proof follows similar arguments as those used in [67, Proposition 2.1], although the presence of varying environment leads to significant changes.

Proposition 1.3.2. *Fix $n\geqslant 1$. Let $\{\dot{Z}_m^{\cdot}:m\leqslant n\}$ and $\{\ddot{Z}_m^{\cdot}:m\leqslant n\}$ be the population size of the size-biased tree and the $X(X-1)$-type size-biased tree up to time n. Then, we have the following decomposition, for each $\lambda\geqslant 0$*

$$\mathbb{E}\left[\exp\left\{-\lambda\ddot{Z}_n^Q\right\}\right]=\mathbb{E}\left[\exp\left\{-\lambda\dot{Z}_n^Q\right\}\right]\sum_{m=0}^{n-1}\mathbb{P}(K_n=m)\mathbb{E}\left[\exp\left\{-\lambda\dot{Z}_{n-(m+1)}^{Q_{m+1}}\right\}\right]g(n,m,\lambda),$$

where the function g is defined as follows

$$g(n,m,\lambda):=\frac{\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\ddot{q}_{m+1}-2]\oplus Q_{m+1}}\right\}\right]}{\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\dot{q}_{m+1}-1]\oplus Q_{m+1}}\right\}\right]},\qquad 0\leqslant m\leqslant n-1,\quad 0\leqslant\lambda. \tag{1.14}$$

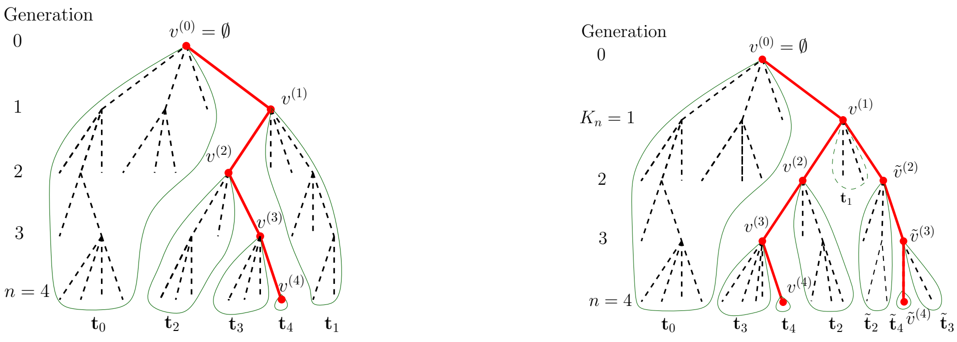
Proof. Let $\dot{\mathbf{T}}$ be a size-biased Galton-Watson tree in environment Q up to time n. We can decompose $\dot{\mathbf{T}}$ into subtrees with roots along the spine $\mathbf{V}$; see Figure 1.2a. More precisely, for every $0\leqslant k\leqslant n$, there is a $v^{(k)}\in\mathbf{V}$ with $|v^{(k)}|=k$ and a random tree $\mathbf{t}_k\in\mathcal{T}$ such that

$$v^{(k)}\,\mathbf{t}_k=\{u\in\dot{\mathbf{T}}:|[\emptyset,u]\cap\mathbf{V}|=k\}\qquad\text{and}\qquad\dot{\mathbf{T}}=\bigsqcup_{k=0}^{n}v^{(k)}\,\mathbf{t}_k,$$

where $\bigsqcup$ denotes the disjoint union. Note that $X_n(\dot{\mathbf{T}})=\sum_{k=0}^{n}X_{n-k}(\mathbf{t}_k)$. In the size-biased tree, each individual along the spine gives birth according to $\dot{q}_{\cdot}$ and one of its offspring is the spine individual in the next generation. Then, it follows that the subtrees $\mathbf{t}_k$, $0\leqslant k\leqslant n$, are independent Galton-Watson trees with environment

$[\dot{q}_{k+1}-1]\oplus Q_{k+1}$. Therefore,

$$\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_n^Q\right\}\right]=\prod_{k=0}^{n}\mathbb{E}\left[\exp\left\{-\lambda Z_{n-k}^{[\dot{q}_{k+1}-1]\oplus Q_{k+1}}\right\}\right], \qquad \lambda\geqslant 0,\ n\in\mathbb{N}_0. \tag{1.15}$$



(a) Size-biased tree

(b) $X(X-1)$-type size-biased tree

Fig. 1.2 Subtrees along the spine(s).

Let $\ddot{\mathbf{T}}$ be a $X(X-1)$-type size-biased Galton-Watson tree up to time n. In a similar way, we can decompose $\ddot{\mathbf{T}}$ in subtrees with roots along the spines; see Figure 1.2b. Denote by $\mathbf{V}$ and $\widetilde{\mathbf{V}}$, the associated spines and recall that

$$K_n=\max\{r<n:\mathbf{V}\overset{r}{=}\widetilde{\mathbf{V}}\}.$$

We can form a partition of $\ddot{\mathbf{T}}$ in the sense that

$$\ddot{\mathbf{T}}=\left(\bigsqcup_{k=0}^{n}v^{(k)}\mathbf{t}_k\right)\bigsqcup\left(\bigsqcup_{k=1+K_n}^{n}\tilde{v}^{(k)}\tilde{\mathbf{t}}_k\right)\quad\text{and}\quad X_n(\ddot{\mathbf{T}})=\sum_{k=0}^{n}X_{n-k}(\mathbf{t}_k)+\sum_{k=1+K_n}^{n}X_{n-k}(\tilde{\mathbf{t}}_k), \tag{1.16}$$

where, for every $0\leqslant k\leqslant K_n$, $v^{(k)}\in\mathbf{V}\cap\widetilde{\mathbf{V}}$ and $\mathbf{t}_k\in\mathcal{T}$ are such that $|v^{(k)}|=k$ and

$$v^{(k)}\,\mathbf{t}_k=\{u\in\ddot{\mathbf{T}}:|[\emptyset,u]\cap(\mathbf{V}\cup\widetilde{\mathbf{V}})|=k\};$$

and, for every $K_n<k\leqslant n$, $v^{(k)}\in\mathbf{V}$, $\tilde{v}^{(k)}\in\widetilde{\mathbf{V}}$ and $\mathbf{t}_k,\tilde{\mathbf{t}}_k\in\mathcal{T}$ satisfy $|v^{(k)}|=k=|\tilde{v}^{(k)}|$,

$$v^{(k)}\,\mathbf{t}_k=\{u\in\ddot{\mathbf{T}}:|[\emptyset,u]\cap\mathbf{V}|=k\}\qquad\text{and}\qquad\tilde{v}^{(k)}\,\tilde{\mathbf{t}}_k=\{u\in\ddot{\mathbf{T}}:|[\emptyset,u]\cap\widetilde{\mathbf{V}}|=k\}.$$

Observe that by the branching property, the subtrees are independent. The spine individual at generation $K_n=m$ has offspring distribution $\ddot{q}_{m+1}$, and from its offspring we select two as the spine individuals in the next generation. Then the subtree $\mathbf{t}_m$ is a Galton-Watson tree with environment $[\ddot{q}_{m+1}-2]\oplus Q_{m+1}$. The other subtrees $\{\mathbf{t}_k : 0 \leqslant k \leqslant n, k \neq m\}$ and $\{\tilde{\mathbf{t}}_k : m < k \leqslant n\}$ are Galton-Watson trees with environment $[\dot{q}_{k+1}-1]\oplus Q_{k+1}$. Therefore, by using the decomposition (1.16), we have

$$\mathbb{E}\left[\exp\left\{-\lambda \ddot{Z}_n^Q\right\}\right] = \sum_{m=0}^{n-1} \mathbb{P}(K_n = m)\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\ddot{q}_{m+1}-2]\oplus Q_{m+1}}\right\}\right]$$
$$\times \prod_{k=0,k\neq m}^{n} \mathbb{E}\left[\exp\left\{-\lambda Z_{n-k}^{[\dot{q}_{k+1}-1]\oplus Q_{k+1}}\right\}\right] \prod_{k=m+1}^{n} \mathbb{E}\left[\exp\left\{-\lambda Z_{n-k}^{[\dot{q}_{k+1}-1]\oplus Q_{k+1}}\right\}\right].$$

Finally, if we apply equation (1.15) for environments Q and Q_{m+1}, we obtain the result. In other words,

$$\mathbb{E}\left[\exp\left\{-\lambda \ddot{Z}_n^Q\right\}\right] = \sum_{m=0}^{n-1} \mathbb{P}(K_n = m)\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\ddot{q}_{m+1}-2]\oplus Q_{m+1}}\right\}\right]$$
$$\times \frac{\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_n^Q\right\}\right]}{\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\dot{q}_{m+1}-1]\oplus Q_{m+1}}\right\}\right]}\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_{n-(m+1)}^{Q_{m+1}}\right\}\right].$$

□

The distribution of the previous processes can be expressed via the generating functions $(f_1, f_2, \ldots)$ associated to $Q = (q_1, q_2, \ldots)$. For each $0 \leqslant m \leqslant n$ and $s \in [0,1]$ we define

$$f_{m,n}(s) := [f_{m+1} \circ \ldots \circ f_n](s),$$

and $f_{n,n}(s) := s$. The generating function of Z_n^Q is equal to $f_{0,n}$. For the others, we note that for every $s \in [0,1]$ and $0 \leqslant m < n$,

$$f'_{m,n}(s) = \prod_{l=m+1}^{n} f'_l(f_{l,n}(s)), \qquad f''_{m,n}(s) = f'_{m,n}(s)^2 \sum_{l=m+1}^{n} \frac{f''_l(f_{l,n}(s))}{f'_l(f_{l,n}(s))^2 \prod_{j=m+1}^{l-1} f'_j(f_{j,n}(s))}, \tag{1.17}$$

where $f'_{n,n}(s) = 1$ and $f''_{n,n}(s) = 0$.

Lemma 1.3.3. *Let $n \geqslant 1$ and Q be a varying environment. Let $(Z_m^{\cdot} : 0 \leqslant m \leqslant n)$, $(\dot{Z}_m^{\cdot} : 0 \leqslant m \leqslant n)$ and $(\ddot{Z}_m^{\cdot} : 0 \leqslant m \leqslant n)$ be a GWVE, a sized-biased GWVE and a $X(X-1)$-type sized-biased GWVE up to time n. Then, for any $0 \leqslant m < n$ and $\lambda \geqslant 0$,*

$$\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\dot{q}_{m+1}-1]\oplus Q_{m+1}}\right\}\right] = \frac{1}{f'_{m+1}(1)} f'_{m+1}(f_{m+1,n}(e^{-\lambda})), \tag{1.18}$$

$$\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\ddot{q}_{m+1}-2]\oplus Q_{m+1}}\right\}\right] = \frac{1}{\nu_{m+1}f'_{m+1}(1)^2}f''_{m+1}(f_{m+1,n}(e^{-\lambda})), \tag{1.19}$$

$$\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_n^Q\right\}\right] = \frac{1}{\mu_n}f'_{0,n}(e^{-\lambda})e^{-\lambda}, \tag{1.20}$$

$$\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_{n-(m+1)}^{Q_{m+1}}\right\}\right] = \frac{\mu_{m+1}}{\mu_n}f'_{m+1,n}(e^{-\lambda})e^{-\lambda}, \tag{1.21}$$

$$\mathbb{E}\left[\exp\left\{-\lambda \ddot{Z}_n^Q\right\}\right] = \frac{1}{\mu_n^2}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} f''_{0,n}(e^{-\lambda})e^{-2\lambda}. \tag{1.22}$$

Proof. Denote by $(g_{m+1}, f_{m+2}, f_{m+3}, \dots)$ the generating functions of the environment $[\dot{q}_{m+1}-1]\oplus Q_{m+1} = ([\dot{q}_{m+1}-1], q_{m+2}, q_{m+3}, \dots)$, where $\dot{q}_{m+1}$ is given in (1.6). Note that,

$$g_{m+1}(s) = \frac{1}{f'_{m+1}(1)}\sum_{k=1}^{\infty} ks^{k-1}q_{m+1}(k) = \frac{1}{f'_{m+1}(1)}f'_{m+1}(s).$$

Then we can deduce (1.18), i.e.

$$\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\dot{q}_{m+1}-1]\oplus Q_{m+1}}\right\}\right] = g_{m+1}\circ f_{m+2}\circ\cdots\circ f_n(e^{-\lambda}) = \frac{1}{f'_{m+1}(1)}f'_{m+1}(f_{m+1,n}(e^{-\lambda})),$$

where we use the probability generating function of a GWVE given in (1.1). The proof of (1.19) follows similar arguments. Recall the definition of $\ddot{q}_{m+1}$ in (1.8). It is enough to see that the generating function of $[\ddot{q}_{m+1}-2]$, denoted by h_{m+1}, is

$$h_{m+1}(s) = \frac{1}{\nu_{m+1}f'_{m+1}(1)^2}\sum_{k=2}^{\infty} k(k-1)s^{k-2}q_{m+1}(k) = \frac{1}{\nu_{m+1}f'_{m+1}(1)^2}f''_{m+1}(s).$$

In order to prove (1.20), note that $\dot{Z}_n^Q$ is a size-biased transform of Z_n^Q. Then, by (1.5)

$$\int_0^{\lambda}\mathbb{E}\left[\exp\left\{-s\dot{Z}_n^Q\right\}\right]\,\mathrm{d}s = \frac{\mathbb{E}\left[1-\exp\left\{-\lambda Z_n^Q\right\}\mid Z_n^Q>0\right]}{\mathbb{E}\left[Z_n^Q\mid Z_n^Q>0\right]} = \frac{\mathbb{E}\left[1-\exp\left\{-\lambda Z_n^Q\right\}\right]}{\mathbb{E}\left[Z_n^Q\right]},$$

for all $\lambda \geqslant 0$. Differentiating the previous equation with respect to λ and using the generating function of Z_n^Q, we obtain

$$\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_n^Q\right\}\right] = \frac{1}{\mu_n}\frac{\mathrm{d}}{\mathrm{d}\lambda}(1-f_{0,n}(e^{-\lambda})) = \frac{1}{\mu_n}f'_{0,n}(e^{-\lambda})e^{-\lambda}.$$

The identity (1.21) is obtained similarly to (1.20) but instead of working with the original environment Q we use the shifted environment Q_{m+1}.

Finally, in order to obtain (1.22) we use the decomposition presented in Proposition 1.3.2

$$\mathbb{E}\left[e^{-\lambda \ddot{Z}_n^Q}\right] = \mathbb{E}\left[e^{-\lambda \dot{Z}_n^Q}\right] \sum_{m=0}^{n-1} \mathbb{P}(K_n = m)\mathbb{E}\left[\exp\left\{-\lambda \dot{Z}_{n-(m+1)}^{Q_{m+1}}\right\}\right]$$
$$\frac{\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\ddot{q}_{m+1}-2]\oplus Q_{m+1}}\right\}\right]}{\mathbb{E}\left[\exp\left\{-\lambda Z_{n-m}^{[\dot{q}_{m+1}-1]\oplus Q_{m+1}}\right\}\right]}.$$

Remember that K_n has distribution (1.7). Hence, substituting the previous Laplace transforms (i.e. equations (1.18),(1.19) and (1.20)) and simplifying, we get

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda \ddot{Z}_n^Q}\right] =& \frac{f'_{0,n}(e^{-\lambda})}{\mu_n} e^{-\lambda} \sum_{m=0}^{n-1} \frac{\nu_{m+1}}{\mu_m} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} \frac{\mu_{m+1}}{\mu_n} \\
& \times \frac{f'_{m+1,n}(e^{-\lambda})e^{-\lambda}}{\nu_{m+1} f'_{m+1}(1)} \frac{f''_{m+1}(f_{m+1,n}(e^{-\lambda}))}{f'_{m+1}(f_{m+1,n}(e^{-\lambda}))} \\
=& e^{-2\lambda} \frac{1}{\mu_n^2} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} f'_{0,n}(e^{-\lambda}) \sum_{m=0}^{n-1} f'_{m+1,n}(e^{-\lambda}) \frac{f''_{m+1}(f_{m+1,n}(e^{-\lambda}))}{f'_{m+1}(f_{m+1,n}(e^{-\lambda}))}.
\end{aligned}$$

Note that for all $s \in [0,1]$ and $0 \leqslant m < n$,

$$f'_{m+1,n}(s) = \prod_{l=m+2}^{n} f'_l(f_{l,n}(s)) = \frac{\prod_{l=1}^{n} f'_l(f_{l,n}(s))}{\prod_{l=1}^{m+1} f'_l(f_{l,n}(s))} = \frac{f'_{0,n}(s)}{f'_{m+1}(f_{m+1,n}(s)) \prod_{l=1}^{m} f'_l(f_{l,n}(s))}.$$

Then,

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda \ddot{Z}_n^Q}\right] =& e^{-2\lambda} \frac{1}{\mu_n^2} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} f'_{0,n}(e^{-\lambda})^2 \\
& \times \sum_{m=0}^{n-1} \frac{f''_{m+1}(f_{m+1,n}(e^{-\lambda}))}{f'_{m+1}(f_{m+1,n}(e^{-\lambda}))^2 \prod_{l=1}^{m} f'_l(f_{l,n}(e^{-\lambda}))} \\
=& e^{-2\lambda} \frac{1}{\mu_n^2} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} f''_{0,n}(e^{-\lambda}).
\end{aligned}$$

This completes the proof. □

The next lemma provides the uniform convergence of the function g defined in (1.14). It is essentially saying that if we start a critical Galton-Watson process with $[\ddot{q}. - 2]$ or $[\dot{q}. - 1]$, the distribution at large times does not change a lot. The reader will find its importance in the next Section. In particular, from Proposition 1.3.2, the lemma gives the precise meaning of equation (1.9).

Lemma 1.3.4. *Suppose that Condition* ($\mathbf{A^*}$) *is fulfilled. Then, for any* $\lambda \geqslant 0$*,*

$$\lim_{n\to\infty} \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \left(1 - g\left(n, m, \frac{s}{a_n^Q}\right)\right) = 0.$$

Proof. By applying Lemma 1.3.3 , we have that for any $s \in [0, \lambda]$ and $0 \leqslant m \leqslant n-1$,

$$g\left(n, m, \frac{s}{a_n^Q}\right) = \frac{f'_{m+1}(1)}{f'_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))} \frac{f''_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))}{f''_{m+1}(1)}.$$

The proof is thus complete as soon as we can show the following uniform convergences

$$\lim_{n\to\infty} \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \left(1 - \frac{f'_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))}{f'_{m+1}(1)}\right) = 0, \tag{1.23}$$

$$\lim_{n\to\infty} \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \left(1 - \frac{f''_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))}{f''_{m+1}(1)}\right) = 0. \tag{1.24}$$

We shall start with (1.23). With the help of the Mean Value Theorem for f'_{m+1} and using that f''_{m+1} is increasing, we obtain

$$0 \leqslant \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \left(1 - \frac{f'_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))}{f'_{m+1}(1)}\right)$$
$$\leqslant \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \frac{f''_{m+1}(1)}{f'_{m+1}(1)} \left(1 - f_{m+1,n}(e^{-s/a_n^Q})\right).$$

Kersting [44, Equation 23] showed that under Condition ($\mathbf{A^*}$), there exists $c > 0$ such that

$$f''_k(1) \leqslant c f'_k(1)(1 + f'_k(1)), \qquad \text{for all } k \geqslant 1. \tag{1.25}$$

Thus

$$\sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} \left(1 - \frac{f'_{m+1}(f_{m+1,n}(e^{-s/a_n^Q}))}{f'_{m+1}(1)}\right)$$
$$\leqslant \sup_{0\leqslant m<n} \sup_{s\in[0,\lambda]} c(1 + f'_{m+1}(1)) \left(1 - f_{m+1,n}(e^{-s/a_n^Q})\right).$$

For similar argument to those given above, using Condition ($\mathbf{A^*}$), and upon an adjustment of the value of the constant, we can get the same upper bound for the left-hand side supremums in (1.24). Therefore, it is enough to prove

$$\lim_{n\to\infty}\sup_{0\leqslant m<n}\sup_{s\in[0,\lambda]}(1+f'_{m+1}(1))(1-\ f_{m+1,n}(e^{-s/a_n^Q}))=0. \tag{1.26}$$

Let $\lambda\geqslant 0$. By the Mean Value Theorem for $f_{m+1,n}$ and using that $f'_{m+1,n}$ is an increasing function, we get for any $0\leqslant s\leqslant\lambda$ and $0\leqslant m<n$

$$0\leqslant(1+f'_{m+1}(1))\left(1-f_{m+1,n}(e^{-s/a_n^Q})\right)\leqslant(1+f'_{m+1}(1))f'_{m+1,n}(1)(1-e^{-s/a_n^Q}).$$

Observe that by Taylor's approximation, $e^{-s/a_n^Q}=1-\frac{s}{a_n^Q}+y_n$ where $y_n\geqslant 0$ is the remainder error term. Then, for $s\in[0,\lambda]$ and $0\leqslant m<n$

$$\begin{aligned}0\leqslant(1+f'_{m+1}(1))\left(1-f_{m+1,n}(e^{-s/a_n^Q})\right) &\leqslant (1+f'_{m+1}(1))\frac{\mu_n}{\mu_{m+1}}\frac{\lambda}{a_n^Q}\\ &= \left(\frac{1}{\mu_{m+1}}+\frac{1}{\mu_m}\right)\frac{\mu_n}{a_n^Q}\lambda. \end{aligned}\tag{1.27}$$

Now, we decompose the left-hand side of (1.26) into two limits where the supremum is taken over two separate sets. Recall that in the critical case, given an $\epsilon>0$ there exists $N>0$ such that $\left(a_k^Q\right)^{-1}\leqslant\epsilon$ for any $k\geqslant N$. Then, we take the two sets as $\{m<N\}$ and $\{N\leqslant m<n\}$. For the first limit, we observe

$$\sup_{0\leqslant m<N}\sup_{s\in[0,\lambda]}(1+f'_{m+1}(1))\left(1-f_{m+1,n}(e^{-\lambda/a_n^Q})\right)\leqslant\frac{\mu_n}{a_n^Q}\lambda\max_{0\leqslant m<N}\left(\frac{1}{\mu_{m+1}}+\frac{1}{\mu_m}\right).$$

By criticality, $\mu_n/a_n^Q\to 0$ as $n\to 0$. Then,

$$\lim_{n\to\infty}\sup_{0\leqslant m<N}\sup_{s\in[0,\lambda]}(1+f'_{m+1}(1))\left(1-f_{m+1,n}(e^{-\lambda/a_n^Q})\right)=0. \tag{1.28}$$

For the second limit, we note that for any $0\leqslant m\leqslant n$,

$$\frac{a_m^Q}{\mu_m}=\frac{1}{2}\sum_{k=0}^{m-1}\frac{\nu_{k+1}}{\mu_k}\leqslant\frac{1}{2}\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}=\frac{a_n^Q}{\mu_n}.$$

Then, by (1.27) and using that $N\leqslant m<n$ we get

$$\sup_{N\leqslant m<n}\sup_{s\in[0,\lambda]}(1+f'_{m+1}(1))\left(1-f_{m+1,n}(e^{-\lambda/a_n^Q})\right)\leqslant\lambda\sup_{N\leqslant m<n}\left(\frac{1}{a_{m+1}^Q}+\frac{1}{a_m^Q}\right)\leqslant 2\epsilon\lambda.$$

Therefore,

$$\lim_{n\to\infty} \sup_{N\leqslant m<n} \sup_{s\in[0,\lambda]} (1+f'_{m+1}(1)) \left(1 - f_{m+1,n}(e^{-\lambda/a_n^Q})\right) = 0,$$

which together with the limit (1.28) gives us (1.26). This concludes the proof. □

1.4 Proof of the main result

As we explained in the outline of the proof, in this chapter we provide a probabilistic argument of a Yaglom-type limit for critical GWVEs. In the previous section we deduced that $\ddot{Z}_n^Q$ is the $Z_n^Q(Z_n^Q-1)$-transform of Z_n^Q and that equation (1.9) holds. Here, we prove the other remaining steps, contained in Proposition 1.4.1 and Proposition 1.4.2. First, we present these propositions. Then, using all the tools that we deduced, we provide the proof for our main result. Finally, we prove the two forthcoming propositions.

Recall the definition of K_n in (1.7). Given the environment Q, we define

$$A_{n,m} := \frac{a_{n-(m+1)}^{Q_{m+1}}}{a_n^Q}, \qquad \text{for } 0 \leqslant m < n.$$

Proposition 1.4.1. *Let Z^Q be a critical GWVE satisfying Condition* (**A***)*. Then*

$$A_{n,K_n} \xrightarrow{(d)} U, \qquad as \;\; n \to \infty,$$

where U is an uniform random variable on $[0,1]$.

Using the previous proposition, we can show the following.

Proposition 1.4.2. *Let $\dot{Z}^Q = \{\dot{Z}_n^Q : n \geqslant 0\}$ be a size-biased GWVE. Then,*

$$(a_n^Q)^{-1}\dot{Z}_n^Q \xrightarrow{(d)} \dot{Y} \qquad as \quad n \to \infty,$$

where $\dot{Y}$ is the size-biased transform of a standard exponential random variable.

We have all the ingredients to prove Yaglom's Theorem under Assumption (**A***).

Proof of Theorem 1.1.1. According to Lemma 1.2.1, in order to deduce Theorem 1.1.1, it is enough to show that $(a_n^Q)^{-1}\dot{Z}_n^Q \xrightarrow{(d)} \dot{Y}$ and $\mathbb{E}\left[(a_n^Q)^{-1} Z_n^Q \mid Z_n^Q > 0\right] \longrightarrow 1$ as

$n \to \infty$, where $\dot{Y}$ is the size-biased transform of an exponential random variable. The first limit holds by Proposition 1.4.2. For the second limit, we observe that

$$\mathbb{E}\left[\frac{Z_n^Q}{a_n^Q}\middle| Z_n^Q > 0\right] = \frac{\mathbb{E}\left[Z_n^Q\right]}{a_n^Q \mathbb{P}\left(Z_n^Q > 0\right)} = \frac{\mu_n}{a_n^Q \mathbb{P}\left(Z_n^Q > 0\right)},$$

which goes to 1 according to (1.4). Therefore, Theorem 1.1.1 holds. □

This chapter is completed as soon as we prove Propositions 1.4.1 and 1.4.2. We start with Proposition 1.4.1.

Proof of Proposition 1.4.1. In order to obtain this result, it is enough to deduce

$$\lim_{n\to\infty} \mathbb{P}\left(A_{n,K_n} \leqslant y\right) = y, \qquad y \in [0,1]. \tag{1.29}$$

Denote by $(\tilde{f}_1, \tilde{f}_2, \dots)$ the generating functions associated with the environment Q_{m+1}. They can be written in terms of the original environment as $\tilde{f}_k = f_{m+1+k}$, for $k \geqslant 1$. Then, by definition

$$\tilde{\mu}_k = f'_{m+2}(1) \cdots f'_{m+1+k}(1) = \frac{\mu_{m+1+k}}{\mu_{m+1}} \qquad \text{and} \qquad \tilde{\nu}_k = \frac{f''_{m+1+k}(1)}{f'_{m+1+k}(1)^2} = \nu_{m+1+k}.$$

Hence,

$$a_{n-(m+1)}^{Q_{m+1}} = \frac{\tilde{\mu}_{n-(m+1)}}{2} \sum_{k=0}^{n-(m+1)-1} \frac{\tilde{\nu}_{k+1}}{\tilde{\mu}_k} = \frac{\mu_n}{2} \sum_{j=m+1}^{n-1} \frac{\nu_{j+1}}{\mu_j},$$

and

$$A_{n,m} = \frac{a_{n-(m+1)}^{Q_{m+1}}}{a_n^Q} = \sum_{j=m+1}^{n-1} \frac{\nu_{j+1}}{\mu_j} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1} = 1 - \sum_{j=0}^{m} \frac{\nu_{j+1}}{\mu_j} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1},$$

where in the last equality, we completed the sum. Then,

$$\begin{aligned} \mathbb{P}\left(A_{n,K_n} = 1 - \sum_{j=0}^{m} \frac{\nu_{j+1}}{\mu_j} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1}\right) &= \mathbb{P}\left(A_{n,K_n} = A_{n,m}\right) = \mathbb{P}\left(K_n = m\right) \\ &= \frac{\nu_{m+1}}{\mu_m} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k}\right)^{-1}. \end{aligned} \tag{1.30}$$

Note that $\{A_{n,m} : m = 0, \dots, n-1\} \subset [0,1]$ is a decreasing sequence with

$A_{n,n-1} = 0$. Then, we can associate it to the partition

$$P^{(n)} = \{0 = \Pi_0^{(n)} < \Pi_1^{(n)} < \ldots < \Pi_{n-1}^{(n)} < \Pi_n^{(n)} = 1\}$$

defined by $\Pi_k^{(n)} = A_{n,n-k-1}$, for any $0 \leqslant k < n$, with $\Pi_n^{(n)} = 1$. The norm of the partition is defined by

$$||P^{(n)}|| = \max_{1\leqslant k\leqslant n} \left\{\Pi_k^{(n)} - \Pi_{k-1}^{(n)}\right\} = \max_{0\leqslant m\leqslant n-1} \left\{\frac{\nu_{m+1}}{\mu_m}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1}\right\}.$$

Since $P^{(n)}$ is a partition, for each $y \in [0,1)$ there exists $l_n := l(y,n) \in \{0,1,\ldots,n-1\}$ such that $\Pi_{l_n}^{(n)} \leqslant y < \Pi_{l_n+1}^{(n)}$. Then, by (1.30)

$$\mathbb{P}\left(A_{n,K_n} \leqslant y\right) = \sum_{k=0}^{l_n} \mathbb{P}\left(A_{n,K_n} = \Pi_k^{(n)}\right) = \sum_{m=n-l_n-1}^{n-1} \frac{\nu_{m+1}}{\mu_m}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} = \Pi_{l_n+1}^{(n)}.$$

It is easy to deduce that in order to prove (1.29), we must prove that $\Pi_{l_n+1}^{(n)} \to y$ as $n \to \infty$. We always choose l_n such that $y \in [\Pi_{l_n}^{(n)}, \Pi_{l_n+1}^{(n)})$. Therefore, it is enough to show that $||P^{(n)}|| \to 0$ as $n \to \infty$.

From inequality (1.25), we see that for each $n \geqslant 1$,

$$\begin{aligned}\frac{\nu_n}{\mu_{n-1}}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} &= \frac{f_n''(1)}{f_n'(1)}\left(\mu_n\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} \leqslant c(1+f_n'(1))\left(\mu_n\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1}\\ &= c\left(\mu_n\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} + c\left(\mu_{n-1}\sum_{k=0}^{n-2}\frac{\nu_{k+1}}{\mu_k} + \nu_n\right)^{-1}.\end{aligned}$$

Since we are in the critical regime and $\nu_n \geqslant 0$ for all $n \geqslant 1$, both summands in the right-hand side of the last equality go to zero as $n \to \infty$. In other words, given $\epsilon > 0$ there exists $N \geqslant 1$ such that

$$\frac{\nu_{m+1}}{\mu_m}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} \leqslant \frac{\nu_{m+1}}{\mu_m}\left(\sum_{k=0}^{m}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} \leqslant \epsilon, \qquad \text{for any } N \leqslant m < n. \tag{1.31}$$

On the other hand, by criticality, for any fixed $m \leqslant N$, there is a $M_m \in \mathbb{N}$ such that

$$\frac{\nu_{m+1}}{\mu_m}\left(\sum_{k=0}^{n-1}\frac{\nu_{k+1}}{\mu_k}\right)^{-1} \leqslant \epsilon, \qquad \text{for any } n \geqslant M_m. \tag{1.32}$$

We define $M = N \vee \max\{M_m : m \leqslant N\}$. Then, by (1.31) and (1.32), for any $n \geqslant M$

$$||P^{(n)}|| = \max_{0 \leqslant m < n} \left\{ \frac{\nu_{m+1}}{\mu_m} \left(\sum_{k=0}^{n-1} \frac{\nu_{k+1}}{\mu_k} \right)^{-1} \right\} \leqslant \epsilon,$$

and the claim holds. $\square$

Now, we present a result whose relevance will become clear in the proof of Proposition 1.4.2. Intuitively, the first statement is an extension of the fact that A_{n,K_n} converges in distribution to U. The limit of $B_3^{(n)}$ is an extension of the fact that $g\left(n, m, s/a_n^Q\right)$ converges uniformly to 1. For the purpose of seeing the intuition in the statement of $B_2^{(n)}$, we normalise $\dot{Z}^{Q_{m+1}}_{n-(m+1)}$ with the correct constant corresponding to the shifted environment. Then, $B_2^{(n)}$ infers that at large times the distributions of the processes $\dot{Z}_n^Q/a_n^Q$ and $\dot{Z}^{Q_{m+1}}_{n-(m+1)}/a^{Q_{m+1}}_{n-(m+1)}$ do not vary much.

Lemma 1.4.3. *Let Q be a varying environment satisfying Condition* $(\mathbf{A^*})$ *and* $\{\dot{Z}_n^{\cdot} : n \geqslant 0\}$ *be a size-biased GWVE. Define*

$$B_1^{(n)} = \int_0^\lambda \left(\mathbb{E}\left[\exp\left\{ -sU \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] - \sum_{m=0}^{n-1} \mathbb{P}(K_n = m) \mathbb{E}\left[\exp\left\{ -sA_{n,m} \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] \right) \mathrm{d}s,$$

$$B_2^{(n)} = \int_0^\lambda \sum_{m=0}^{n-1} \mathbb{P}(K_n = m) \left(\mathbb{E}\left[\exp\left\{ -sA_{n,m} \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] - \mathbb{E}\left[\exp\left\{ -s \frac{\dot{Z}^{Q_{m+1}}_{n-(m+1)}}{a_n^Q} \right\} \right] \right) \mathrm{d}s,$$

$$B_3^{(n)} = \int_0^\lambda \sum_{m=0}^{n-1} \mathbb{P}(K_n = m) \mathbb{E}\left[\exp\left\{ -s \frac{\dot{Z}^{Q_{m+1}}_{n-(m+1)}}{a_n^Q} \right\} \right] \left(1 - g\left(n, m, \frac{s}{a_n^Q} \right) \right) \mathrm{d}s,$$

where U is an uniform random variable on $[0,1]$ independent of $\dot{Z}^Q$. Then,

$$\limsup_{n\to\infty} |B_1^{(n)}| = \limsup_{n\to\infty} |B_2^{(n)}| = \limsup_{n\to\infty} |B_3^{(n)}| = 0.$$

Proof. We start with $B_1^{(n)}$. Recall the partition

$$P^{(n)} = \{\Pi_0^{(n)} < \Pi_1^{(n)} < \ldots < \Pi_{n-1}^{(n)} < \Pi_n^{(n)}\}$$

given in the proof of Proposition 1.4.1 and that $\mathbb{P}(K_n = m) = \Pi^{(n)}_{n-m} - \Pi^{(n)}_{n-m-1}$. Then

$$\begin{aligned} b_1^{(n)}(s) :=& \mathbb{E}\left[\exp\left\{ -sU \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] - \sum_{m=0}^{n-1} \mathbb{P}(K_n = m) \mathbb{E}\left[\exp\left\{ -sA_{n,m} \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] \\ =& \int_0^1 \mathbb{E}\left[\exp\left\{ -su \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] \mathrm{d}u - \sum_{m=0}^{n-1} (\Pi^{(n)}_{n-m} - \Pi^{(n)}_{n-m-1}) \mathbb{E}\left[\exp\left\{ -sA_{n,m} \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right]. \end{aligned}$$

By decomposing $[0,1]$ into the subintervals $[\Pi^{(n)}_{n-m-1},\Pi^{(n)}_{n-m}]$, $m=0,\dots,n-1$, we get

$$b_1^{(n)}(s)=\sum_{m=0}^{n-1}\int_{\Pi^{(n)}_{n-m-1}}^{\Pi^{(n)}_{n-m}}\mathbb{E}\left[\exp\left\{-su\frac{\dot{Z}_n^Q}{a_n^Q}\right\}\right]\mathrm{d}u-\sum_{m=0}^{n-1}\int_{\Pi^{(n)}_{n-m-1}}^{\Pi^{(n)}_{n-m}}\mathbb{E}\left[\exp\left\{-sA_{n,m}\frac{\dot{Z}_n^Q}{a_n^Q}\right\}\right]\mathrm{d}u.$$

Now, by Lemma 1.3.3, the Laplace transform of $\dot{Z}_n^Q$ can be expressed in terms of $f'_{0,n}$. Since $x\mapsto f'_{0,n}(e^{-\lambda x})e^{-\lambda x}$ is a decreasing function, and $u,A_{n,m}\in[\Pi^{(n)}_{n-m-1},\Pi^{(n)}_{n-m}]$ for $m=0,\dots,n-1$, we deduce

$$\begin{aligned}
|b_1^{(n)}(s)|\leqslant&\frac{1}{\mu_n}\sum_{m=0}^{n-1}\int_{\Pi^{(n)}_{n-m-1}}^{\Pi^{(n)}_{n-m}}\left|f'_{0,n}\left(e^{-su/a_n^Q}\right)e^{-su/a_n^Q}-f'_{0,n}\left(e^{-sA_{n,m}/a_n^Q}\right)e^{-sA_{n,m}/a_n^Q}\right|\mathrm{d}u\\
\leqslant&\frac{1}{\mu_n}\sum_{m=0}^{n-1}(\Pi^{(n)}_{n-m}-\Pi^{(n)}_{n-m-1})\\
&\times\left(f'_{0,n}\left(e^{-s\Pi^{(n)}_{n-m-1}/a_n^Q}\right)e^{-s\Pi^{(n)}_{n-m-1}/a_n^Q}-f'_{0,n}\left(e^{-s\Pi^{(n)}_{n-m}/a_n^Q}\right)e^{-s\Pi^{(n)}_{n-m}/a_n^Q}\right).
\end{aligned}$$

The last sum can be bounded by the norm of the partition multiplied by a telescopic sum with $\Pi_0^{(n)}=0$ and $\Pi_n^{(n)}=1$. Therefore

$$|b_1^{(n)}(s)|\leqslant\frac{1}{\mu_n}||P^{(n)}||\left(f'_{0,n}(1)-f'_{0,n}\left(e^{-s/a_n^Q}\right)e^{-s/a_n^Q}\right)\leqslant\frac{1}{\mu_n}f'_{0,n}(1)||P^{(n)}||=||P^{(n)}||.$$

Since the norm of the partition goes to zero as $n\to\infty$ (see the proof of Proposition 1.4.1), we get the result for $B_1^{(n)}$,

$$\limsup_{n\to\infty}|B_1^{(n)}|\leqslant\limsup_{n\to\infty}\int_0^\lambda|b_1^{(n)}(s)|\,\mathrm{d}s\leqslant\limsup_{n\to\infty}\lambda||P^{(n)}||=0.$$

Now we deal with $B_2^{(n)}$. By Lemma 1.3.3, the Laplace transform of $\dot{Z}_n^Q$ and $\dot{Z}^{Q_{m+1}}_{n-m-1}$ can be expressed in terms of $f'_{0,n}$ and $f'_{m+1,n}$, respectively. Then,

$$\begin{aligned}
b_2^{(n,m)}(s):=&\mathbb{E}\left[\exp\left\{-sA_{n,m}\frac{\dot{Z}_n^Q}{a_n^Q}\right\}\right]-\mathbb{E}\left[\exp\left\{-s\frac{\dot{Z}^{Q_{m+1}}_{n-(m+1)}}{a_n^Q}\right\}\right]\\
=&\frac{1}{\mu_n}f'_{0,n}\left(e^{-sA_{n,m}/a_n^Q}\right)e^{-sA_{n,m}/a_n^Q}-\frac{\mu_{m+1}}{\mu_n}f'_{m+1,n}\left(e^{-s/a_n^Q}\right)e^{-s/a_n^Q}.
\end{aligned}$$

Using first the Fundamental Theorem of Calculus and then the Mean Value Theorem in the functions $f_{0,n}$ and $f_{m+1,n}$, we deduce that

$$\begin{aligned}
&\int_0^\lambda b_2^{(n,m)}(s)\,\mathrm{d}s \\
&= \frac{1}{\mu_n}\left(\frac{a_n^Q}{A_{n,m}}\left(f_{0,n}(1)-f_{0,n}\left(e^{-\lambda A_{n,m}/a_n^Q}\right)\right)-\mu_{m+1}a_n^Q\left(f_{m+1,n}(1)-f_{m+1,n}\left(e^{-\lambda/a_n^Q}\right)\right)\right) \\
&= \frac{1}{\mu_n}\left(\frac{a_n^Q}{A_{n,m}}\left(1-e^{-\lambda A_{n,m}/a_n^Q}\right)f'_{0,n}(\xi)-\mu_{m+1}a_n^Q\left(1-e^{-\lambda/a_n^Q}\right)f'_{m+1,n}(\eta)\right),
\end{aligned} \tag{1.33}$$

where $\xi \in \left(e^{-\lambda A_{n,m}/a_n^Q},1\right)$ and $\eta \in \left(e^{-\lambda/a_n^Q},1\right)$. Now, we shall find $\widehat{B}_2^{(n)}$ and $\widetilde{B}_2^{(n)}$ such that $\widehat{B}_2^{(n)} \to 0$, $\widetilde{B}_2^{(n)} \to 0$ as $n \to \infty$ and

$$B_2^{(n)} = \sum_{m=0}^{n-1}\mathbb{P}(K_n=m)\int_0^\lambda b_2^{(n,m)}(s)\,\mathrm{d}s \in \left[\widehat{B}_2^{(n)},\widetilde{B}_2^{(n)}\right]. \tag{1.34}$$

The fact that $f'_{0,m}$ and $f'_{m,n}$ are increasing functions and (1.33) imply the following lower bound

$$\begin{aligned}
B_2^{(n)} \geqslant \widehat{B}_2^{(n)} := \sum_{m=0}^{n-1}\frac{\mathbb{P}(K_n=m)}{\mu_n}&\left(\frac{a_n^Q}{A_{n,m}}\left(1-e^{-\lambda A_{n,m}/a_n^Q}\right)f'_{0,n}(e^{-\lambda A_{n,m}/a_n^Q})\right. \\
&\left.-\mu_{m+1}a_n^Q\left(1-e^{-\lambda/a_n^Q}\right)f'_{m+1,n}(1)\right).
\end{aligned}$$

Since $\mathbb{P}(K_n=m)=\mathbb{P}(A_{n,K_n}=A_{n,m})$ and $\mu_{m+1}f'_{m+1,n}(1)=\mu_n$, we have

$$\widehat{B}_2^{(n)} = \mathbb{E}\left[\frac{a_n^Q}{A_{n,K_n}}\left(1-e^{-\lambda A_{n,K_n}/a_n^Q}\right)\frac{f'_{0,n}\left(e^{-\lambda A_{n,K_n}/a_n^Q}\right)}{f'_{0,n}(1)}\right]-a_n^Q\left(1-e^{-\lambda/a_n^Q}\right).$$

On the other hand, for the upper bound, observe that $\mu_{m+1}=f'_1(1)\cdots f'_{m+1}(1)$, that $f_{l,n}(e^{-\lambda/a_n^Q})\leqslant 1$ for any $1\leqslant l\leqslant m+1$, and that f'_l is an increasing function, then

$$\mu_{m+1}f'_{m+1,n}(e^{-\lambda/a_n^Q}) \geqslant \prod_{l=1}^{m+1}f'_l\left(f_{l,n}\left(e^{-\lambda/a_n^Q}\right)\right)f'_{m+1,n}\left(e^{-\lambda/a_n^Q}\right)=f'_{0,n}\left(e^{-\lambda/a_n^Q}\right),$$

where in the equality we use (1.17). By (1.33) and (1.34), the previous inequality implies that

$$\begin{aligned}B_2^{(n)} \leqslant \widetilde{B}_2^{(n)} :=& \sum_{m=0}^{n-1} \frac{\mathbb{P}(K_n=m)}{\mu_n}\left(\frac{a_n^Q}{A_{n,m}}\left(1-e^{-\lambda A_{n,m}/a_n^Q}\right) f'_{0,n}(1)\right.\\ &\left. -a_n^Q\left(1-e^{-\lambda/a_n^Q}\right) f'_{0,n}(e^{-\lambda/a_n^Q})\right)\\ =&\mathbb{E}\left[\frac{a_n^Q}{A_{n,K_n}}\left(1-e^{-\lambda A_{n,K_n}/a_n^Q}\right)\right] - a_n^Q\left(1-e^{-\lambda/a_n^Q}\right)\frac{f'_{0,n}\left(e^{-\lambda/a_n^Q}\right)}{f'_{0,n}(1)}.\end{aligned}$$

Then, (1.34) holds. Now, we show that the limit of $\widehat{B}_2^{(n)}$ and $\widetilde{B}_2^{(n)}$ is zero as $n\to\infty$. Recall that $0 \leqslant A_{n,K_n} \leqslant 1$ and $a_n^Q \to \infty$ as $n\to\infty$, then as $n\to\infty$

$$a_n^Q\left(1-e^{-\frac{\lambda}{a_n^Q}}\right) \to \lambda, \qquad \frac{f'_{0,n}\left(e^{-\lambda/a_n^Q}\right)}{f'_{0,n}(1)} \to 1,$$

and

$$\frac{a_n^Q}{A_{n,K_n}}\left(1-e^{-\lambda A_{n,K_n}/a_n^Q}\right) \to \lambda, \qquad \text{and} \qquad \frac{f'_{0,n}\left(e^{-\lambda A_{n,K_n}/a_n^Q}\right)}{f'_{0,n}(1)} \to 1 \qquad \text{a.s.}$$

By Dominated Convergence Theorem, we have that $\widehat{B}_2^{(n)} \to 0$ and $\widetilde{B}_2^{(n)} \to 0$ as $n\to\infty$. Therefore, $B_2^{(n)}$ has the same behaviour. Since the limit is zero we also have that $|B_2^{(n)}| \to 0$ as $n\to\infty$.

Finally, we deal with $B_3^{(n)}$. Given an $\epsilon > 0$, by Lemma 1.3.4, there exists $M > 0$ such that for $n \geqslant M$, $0 \leqslant s \leqslant \lambda$ and $0 \leqslant m < n$

$$\left|g\left(n, m, \frac{s}{a_n^Q}\right) - 1\right| \leqslant \epsilon.$$

Hence, for $n \geqslant M$,

$$|B_3^{(n)}| \leqslant \int_0^\lambda \sum_{m=0}^{n-1} \mathbb{P}(K_n=m)\left|g\left(n, m, \frac{s}{a_n^Q}\right) - 1\right| \mathrm{d}s \leqslant \epsilon\lambda.$$

Since ϵ is arbitrary, we get that $\limsup_{n\to\infty} |B_3^{(n)}| = 0$. $\square$

For the proof of Proposition 1.4.2, we need the following two lemmas, which the reader can find in [67, Lemma 3.1 and Lemma 3.2]. The first lemma compares the generating functions of two variables with the generating functions of their size-biased transforms. The second lemma is similar to Grönwall's Lemma.

Lemma 1.4.4. *Let* X *and* W *be two non-negative random variables with mean* μ. *Let* F *and* G *be functions such that* $\mathbb{E}\left[e^{-\lambda\dot{X}}\right] = \mathbb{E}\left[e^{-\lambda X}\right]F(\lambda)$ *and* $\mathbb{E}\left[e^{-\lambda\dot{W}}\right] = \mathbb{E}[e^{-\lambda W}]G(\lambda)$, *where* $\dot{X}$ *and* $\dot{W}$ *are the size-biased transforms of* X *and* W. *Then,*

$$\left|\mathbb{E}\left[e^{-\lambda X}\right] - \mathbb{E}\left[e^{-\lambda W}\right]\right| \leqslant \mu\left|\int_0^\lambda (F(s)-G(s))\,\mathrm{d}s\right|, \qquad \lambda \geqslant 0.$$

Lemma 1.4.5. *Suppose that a non-negative bounded function* F *on* $[0,\infty)$ *and a constant* $c>0$ *satisfy*

$$F(\lambda) \leqslant c\int_0^1 \mathrm{d}u \int_0^\lambda F(us)\,\mathrm{d}s, \qquad \textit{for } \lambda \geqslant 0.$$

Then $F \equiv 0$.

Finally, we present the last proof in this chapter.

Proof of Proposition 1.4.2. We define the bounded function

$$M(\lambda) = \limsup_{n\to\infty}\left|\mathbb{E}[e^{-\lambda\dot{Y}}] - \mathbb{E}\left[\exp\left\{-\lambda\frac{\dot{Z}_n^Q}{a_n^Q}\right\}\right]\right|, \qquad \text{for } \lambda \geqslant 0.$$

We will use Lemma 1.4.4 with $X = \dot{Y}$ and $W = (\dot{Z}_n^Q - 1)/a_n^Q$. Since Y is an exponential random variable and (1.2) holds, we get

$$\mathbb{E}[\dot{Y}] = 2 = \mathbb{E}\left[\frac{\dot{Z}_n^Q - 1}{a_n^Q}\right].$$

Thanks to the characterisation (1.11), we may choose $F(\lambda) = \mathbb{E}[e^{-\lambda U\dot{Y}}]$, where U is an uniform variable on $[0,1]$ independent of $\dot{Y}$. Then, by Proposition 1.3.2, we have

$$G(\lambda) = \sum_{m=0}^{n-1}\mathbb{P}(K_n = m)g\left(n,m,\frac{\lambda}{a_n^Q}\right)\mathbb{E}\left[\exp\left\{-\lambda\frac{\dot{Z}^{Q_{m+1}}_{n-(m+1)}}{a_n^Q}\right\}\right],$$

where g is given in (1.14). Hence, by Lemma 1.4.4 and the triangle inequality,

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{-\lambda\dot{Y}}\right] - \mathbb{E}\left[\exp\left\{-\lambda\frac{\dot{Z}_n^Q - 1}{a_n^Q}\right\}\right]\right| \\
&\leqslant 2\left|\int_0^\lambda\left(\mathbb{E}[e^{-sU\dot{Y}}] - \sum_{m=0}^{n-1}\mathbb{P}(K_n=m)g\left(n,m,\frac{s}{a_n^Q}\right)\mathbb{E}\left[\exp\left\{-s\frac{\dot{Z}^{Q_{m+1}}_{n-(m+1)}}{a_n^Q}\right\}\right]\right)\mathrm{d}s\right| \\
&\leqslant 2\left(|B_1^{(n)}| + |B_2^{(n)}| + |B_3^{(n)}| + |B_4^{(n)}|\right),
\end{aligned}$$

where $B_1^{(n)}$, $B_2^{(n)}$ and $B_3^{(n)}$ are defined in Lemma 1.4.3 and

$$B_4^{(n)} = \int_0^\lambda \left(\mathbb{E}\left[e^{-sU\dot{Y}} \right] - \mathbb{E}\left[\exp\left\{ -sU \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] \right) \mathrm{d}s,$$

with U being a uniform random variable on $[0,1]$ independent of $\dot{Y}$ and $\dot{Z}^Q$. Then, by Lemma 1.4.3 and the Dominated Convergence Theorem, we obtain

$$M(\lambda) \leqslant 2 \limsup_{n\to\infty} \int_0^\lambda \left| \mathbb{E}\left[e^{-sU\dot{Y}} \right] - \mathbb{E}\left[\exp\left\{ -sU \frac{\dot{Z}_n^Q}{a_n^Q} \right\} \right] \right| \mathrm{d}s \leqslant 2 \int_0^\lambda \int_0^1 M(us)\, \mathrm{d}u\, \mathrm{d}s.$$

By Lemma 1.4.5, $M \equiv 0$ which implies that $\dot{Z}_n^Q / a_n^Q$ converges weakly to $\dot{Y}$. $\square$

Part II

Contact processes with fitness on Galton-Watson trees

Chapter 2

The contact process with fitness on Galton-Watson trees

The contact process is a simple model for the spread of an infection in a structured population. We consider a variant of this process on Galton-Watson trees, where vertices are equipped with a random fitness representing inhomogeneities among individuals. In this chapter, we establish conditions under which the contact process with fitness on Galton-Watson trees exhibits a phase transition. We prove that if the distribution of the product of the offspring and the fitness has exponential tails then the survival threshold is strictly positive. Further, we show that, under certain conditions on either the fitness distribution or the offspring distribution, there is no phase transition and the process survives with positive probability for any choice of the infection parameter. A similar dichotomy is known for the contact process on a Galton-Watson tree. However, we see that the introduction of fitness means that we have to take into account the combined effect of fitness and offspring distribution to decide which scenario occurs. The chapter is organised as follows. In Section 2.1, we give an introduction to the contact processes. In Section 2.2, we define the contact process with fitness and we state our main results. Further, in this section we give an overview of the proofs of the main results. In Section 2.3, we present some useful properties of the contact process. Section 2.4 is devoted to the proof of our first main result. Section 2.5 is devoted to some results regarding to contact process with fitness on finite stars. Finally, in Section 2.6, we prove our second main result.

2.1 Introduction

The contact process on a graph is a model that describes the spread of an infection in a population, and it is among the most studied particle systems. The model is described informally as follows. The vertices of the graph represent individuals that are susceptible to the infection and the edges depict the connections between them. We assume that infection and recovery events happen independently from vertex to vertex. The recovery rate is constant and the intensity of infection depends on a constant $\lambda > 0$. The contact process is sometimes referred to as the susceptible–infected–susceptible (SIS) epidemic model.

The behaviour of the contact process depends on the infection parameter λ. Therefore it is natural to ask when there exists a critical value of λ for which the contact process exhibits a phase transition. For an infinite rooted graph, there are two critical values of interest $0 \leqslant \lambda_1 \leqslant \lambda_2$, which determine different regimes where the contact process exhibit extinction, weak survival or strong survival. In others words, in the *extinction* phase, for $\lambda \in (0, \lambda_1)$, the infection becomes extinct in finite time almost surely. In the *weak survival* phase, when $\lambda \in (\lambda_1, \lambda_2)$, the infection survives forever with positive probability, and the root is infected finitely many times almost surely. Finally, in the *strong survival* phase, for $\lambda \in (\lambda_2, \infty)$, the infection also survives forever with positive probability, however in this regime the root is infected infinitely many times with positive probability. On the other hand, for a finite graph, the infection dies out almost surely in finite time. An interesting question here is e.g. how long the process survives expressed in terms of the size of the graph. See for instance [54] for a general introduction to the topic.

There are several works in the literature where phase transitions have been studied for different graphs. The first known work is by Harris [37], he showed that the contact process on the integer lattice $\mathbb{Z}^d$ (for any $d \geqslant 1$) does not exhibit a weak survival phase, that is to say, $0 < \lambda_1(\mathbb{Z}^d) = \lambda_2(\mathbb{Z}^d) < \infty$. In other words, if $\lambda < \lambda_1$ the contact process on $\mathbb{Z}^d$, started with the origin infected, dies out with probability 1 and survives forever with positive probability if $\lambda > \lambda_1$. For a complete account, the reader is referred to the book of Liggett [54].

One of the first analysis on graphs other than $\mathbb{Z}^d$ was carried out by Pemantle [64] on infinite d-ary tree $\mathbb{T}_d$. He showed that the contact process on $\mathbb{T}_d$ for $d \geqslant 3$ with the root initially infected satisfies $0 < \lambda_1(\mathbb{T}_d) < \lambda_2(\mathbb{T}_d) < \infty$. This result was later extended to the case $d = 2$ by Liggett [53]. Afterwards Stacey [69] gave a short proof which works for all $d \geqslant 2$. The contact process has also been studied on certain non-homogeneous classes of graphs. Chatterjee and Durret [20] considered the contact process on models of power law random graphs and studied the critical value. To be more specific, consider a power law random graph such that the degree of a vertex is k with probability $p_k \sim Ck^{-\alpha}$, as $k \to \infty$ for some constant $C > 0$. The authors in [20] showed that the critical value of the infection parameter is zero for any $\alpha > 3$ contradicting mean-field calculations as previously obtained in [61, 62].

Much less is known about the contact process on random trees. Huang and Durret [41] showed that for the contact process on Galton-Watson trees with the root initially infected, the critical value for local survival is $\lambda_2 = 0$ if the offspring distribution $\mathscr{L}(\xi)$ is subexponential, i.e., if $\mathbb{E}[e^{c\xi}] = \infty$ for all $c > 0$. Shortly afterwards, Bhamidi et al. [13] proved that on Galton-Watson trees, $\lambda_1 > 0$ if the offspring distribution $\mathscr{L}(\xi)$ has an exponential tail, i.e., if $\mathbb{E}[e^{c\xi}] < \infty$ for some $c > 0$. These two results give a complete characterisation on the existence of extinction phase on Galton-Watson trees.

A natural generalization of the contact process is to introduce inhomogeneity into the graph by associating a random fitness to each vertex that influences how likely the vertex is to receive and to pass on the infection. Peterson [66] introduced the contact process on a (deterministic) complete graph with random vertex-dependent infection rates. In this model, the rate at which the infection travels along the edge depends on the vertex weights. More precisely, under a second moment assumption on the weights, he proved that there is a phase transition at $\lambda_c > 0$ such that for $\lambda < \lambda_c$ the contact process dies out in logarithmic time, and for $\lambda > \lambda_c$ the contact process lives for an exponential amount of time. The way that the infection rates were chosen by Peterson was inspired by inhomogeneous random graphs as introduced by Chung and Lu [23], where, given a sequence of vertex weights, the probability that there is an edge connecting two vertices is proportional to the product of the weights. Xue [72, 74] studied the contact process with random vertex weights with bounded support on oriented lattices. In particular, the author investigates in [72] the asymptotic behaviour of the critical value when the lattice dimension grows. Later, Pan et al. [60] extended his result to the case of regular trees. The reader is also referred to [71, 73] for further results about the contact process with random weights and bounded support on regular graphs.

In this chapter, we are interested in understanding the interplay between the inhomogeneous contact process as considered by [66] with a more structured graph and therefore consider the model on a Galton-Watson tree. We focus on Galton-Watson trees, since these can be often used to describe the local geometry of random graphs and standard techniques should apply to translate our results to random graphs. A natural interest is then to study the phase diagram of this model and to understand how the extra randomness changes the characterisation of whether a phase transition occurs or not.

The main contribution of this chapter is to understand the phase transition of the contact process with fitness on Galton-Watson trees. Our first result shows the existence of a phase transition when the distribution of the product of offspring and fitness has exponential tails. In other words, if $\mathcal{F}$ the fitness associate to a vertex, we prove that if $\mathscr{L}(\xi\mathcal{F})$ has exponential tails, then the survival threshold is strictly positive. The second results tell us that, under certain condition on either the offspring distribution or on the fitness distribution, we have that the process survives strongly (with positive probability).

In our setting, the fitness has a significant effect on the behaviour of the model as a whole. For instance, if we consider the standard contact process on a Galton-Watson tree, where the offspring distribution has exponential tails, then, as mentioned earlier, the process exhibits the phase transition, so in particular the process dies out for small enough λ. However, if the random i.i.d. fitness values is sufficiently heavy-tailed, then irrespectively how light the tails of the offspring distribution are (as long as it has unbounded support), then the process no longer has a phase of extinction. In other words, the presence of fitness guarantees that the infection survives forever with positive probability regardless of the value of λ. Similarly, just the fact that there is a random fitness with unbounded support (without any tail assumptions) means that certain offspring distribution with lighter tails lead to the lack of a phase transition, even if the classical model with the same offspring distribution would exhibit a subcritical phase.

2.2 Definitions and main results

In this section, we briefly introduce the primary notions and discuss the main results in the manuscript.

Let us denote by $\mathcal{T} \sim \mathbf{GW}(\xi)$ the Galton-Watson tree rooted at ρ with offspring distribution $\mathscr{L}(\xi)$. We assume that $\mu = \mathbb{E}[\xi] > 1$, which makes $\mathcal{T}$ survive forever with positive probability. Denote by $V(\mathcal{T})$ the set of vertices in $\mathcal{T}$. We equip each vertex $v \in V(\mathcal{T})$ of the tree with a random initial fitness. More precisely, let $\mathbb{F}(\mathcal{T}) := (\mathcal{F}_v)_{v\in V(\mathcal{T})}$ be a sequence of i.i.d copies of a non-negative random variable. For example, we assign fitness values $\mathcal{F}_\rho, \mathcal{F}_{v_1}, \mathcal{F}_{v_2} \dots$ to the vertices $\rho, v_1, v_2, \dots$, respectively. We denote by $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ this weighted version tree. When the context is clear we simply denote the weighted tree by $\mathcal{T}$.

Definition 2.2.1. *Let* $\mathbb{F}(\mathcal{T}) := (\mathcal{F}_v)_{v\in V(\mathcal{T})}$ *be a sequence of i.i.d. copies of a random variable* $\mathcal{F}$ *taking values in* $[1,\infty)$. *The* inhomogeneous contact process *on* $(\mathcal{T},\mathbb{F}(\mathcal{T}))$ *is a continuous-time Markov chain on the state space* $\{0,1\}^{V(\mathcal{T})}$, *where a vertex is either infected (state 1) or healthy (state 0). We denote the process by*

$$(X_t) \sim \mathbf{CP}\Big(\mathcal{T};\mathbf{1}_A\Big),$$

where $\mathbf{1}_A$ *is the initial configuration, where the vertices in* $A \subset V(\mathcal{T})$ *are initially infected. Given the fitness values and* $\lambda > 0$, *the process evolves according to the following rules:*

- *For each* $v \in V(\mathcal{T})$ *such that* $X_t(v) = 1$, *the process* X_t *becomes* $X_t - \mathbf{1}_v$ *at rate 1.*
- *For each* $v \in V(\mathcal{T})$ *such that* $X_t(v) = 0$, *the process* X_t *becomes* $X_t + \mathbf{1}_v$ *at rate*

  $$\lambda \sum_{v\sim v'} \mathcal{F}_{v'}\mathcal{F}_v X_t(v'),$$

  where $\mathcal{F}_v$ *and* $\mathcal{F}_{v'}$ *are the fitness values associated to* v *and* v'. *The notation* $v \sim v'$ *means that vertices* v *and* v' *are connected by an edge in* $\mathcal{T}$.

Notation: We use the notation $\mathbf{0}$ for the all-healthy state, i.e., $\mathbf{0} = \mathbf{1}_\emptyset$. We also often identify any state $\{0,1\}^{V(\mathcal{T})}$ with the subset of $V(\mathcal{T})$ consisting of the vertices that have state 1 (i.e. the infected vertices). For example, when we write $\rho \in X_t$, it means that the root of the tree is infected at time t. We use the conditional probability measure $\mathbb{P}_{\mathcal{T},\mathbb{F}}(\cdot) := \mathbb{P}\Big(\cdot \mid \mathcal{T},\mathbb{F}(\mathcal{T})\Big)$ with associated expectation operator $\mathbb{E}_{\mathcal{T},\mathbb{F}}[\cdot] := \mathbb{E}\Big[\cdot \mid \mathcal{T},\mathbb{F}(\mathcal{T})\Big]$. For any set A, we write $|A|$ to denote its cardinality.

Now, we define the critical values for the infection parameter λ. Given the tree $(\mathcal{T},\mathbb{F}(\mathcal{T}))$, we define the threshold between extinction and weak survival by

$$\lambda_1(\mathcal{T},\mathbb{F}(\mathcal{T})) := \inf\Big\{\lambda :\ \mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(X_t \neq \mathbf{0} \text{ for all } t \geqslant 0\Big) > 0\Big\},$$

and the weak-strong survival threshold by

$$\lambda_2(\mathcal{T},\mathbb{F}(\mathcal{T})) := \inf\Big\{\lambda :\ \liminf_{t\to\infty} \mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(\rho \in X_t\Big) > 0\Big\}.$$

By using the same arguments as those used in Proposition 3.1 in Pemantle [64], we see that $\lambda_1(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ and $\lambda_2(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ are constant for almost every $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ conditioned on $|\mathcal{T}| = \infty$. We denote by λ_1 and λ_2 these two constants.

Let $(\widehat{X_t})$ denote the contact process with constant fitness. More precisely, infected individual v recovers from its infection after an exponential time with mean 1, independently of the status of its neighbours, while healthy individual v becomes infected at a rate that is proportional to the number of infected neighbours, i.e., a rate

$$\lambda \sum_{v \sim v'} \widehat{X}_t(v'),$$

where the sum is over the neighbours v' of v. We may thus identify $(\widehat{X_t})$ with a particular case of our model where we take $\mathcal{F} \equiv 1$. Further, note that the requirement $\mathcal{F} \geqslant 1$ a.s. and the monotonicity of the contact process ensures that $\widehat{X_t} \subset X_t$ for all $t \geqslant 0$, if we start with the same initial configuration (see Section 2.3 for further details regarding to the monotonicity property). According to [41, Theorem 1.4] and [13, Theorem 1], the process $(\widehat{X_t})$ always survives (at least weakly) and thus also (X_t), for all sufficiently large λ.

Throughout this chapter, we shall suppose that, $\mu = \mathbb{E}[\xi] < \infty$. This condition guarantees that the set of infected vertices at every time is finite a.s. (see Theorem 2.3.2).

Theorema 2.2.1. *Consider the inhomogeneous contact process on the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. Suppose that only the root of the tree is initially infected. We assume that the distribution of the product of ξ and the fitness $\mathcal{F}$ has exponential tails, i.e.,*

$$\mathbb{E}\Big[e^{c\xi\mathcal{F}}\Big] = M < \infty \quad \textit{for some} \quad c, M > 0. \tag{\textbf{A}}$$

Then there exists $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$, the process dies out almost surely.

Remark 2.2.1. Note that Assumption (**A**) implies that the distributions of ξ as well as that of $\mathcal{F}$ have exponential tails. Indeed, for $c > 0$ as in the assumption and $n \geqslant 1$ we have

$$\begin{aligned} \mathbb{E}\Big[e^{c\mathcal{F}}\mathbf{1}_{\{\mathcal{F}\leqslant n\}}\Big] &= \mathbb{E}\Big[e^{c\mathcal{F}}\mathbf{1}_{\{\mathcal{F}\leqslant n\}}\mathbf{1}_{\{\xi\geqslant 1\}}\Big] + \mathbb{E}\Big[e^{c\mathcal{F}}\mathbf{1}_{\{\mathcal{F}\leqslant n\}}\mathbf{1}_{\{\xi=0\}}\Big] \\ &\leqslant \mathbb{E}\Big[e^{c\mathcal{F}\xi}\mathbf{1}_{\{\mathcal{F}\leqslant n\}}\Big] + \mathbb{P}(\xi = 0)\mathbb{E}\Big[e^{c\mathcal{F}}\mathbf{1}_{\{\mathcal{F}\leqslant n\}}\Big]. \end{aligned}$$

Now taking limits, as $n \to \infty$, and appealing to the Monotone Convergence Theorem, we deduce that

$$\mathbb{E}\Big[e^{c\mathcal{F}}\Big] \leqslant \frac{\mathbb{E}\Big[e^{c\mathcal{F}\xi}\Big]}{1-\mathbb{P}(\xi=0)} < \infty.$$

Similarly, we have that $\mathbb{E}\Big[e^{c\xi}\Big] < \infty$ for some $c > 0$. Intuitively, we need at least exponential tails on the degree distribution as well as in the fitness distribution in order to control the number of vertices with high degree and vertices with high fitness. If we control such vertices the infection will persist for a short time around them and thus extinction will be almost surely.

Furthermore, observe that in general assuming exponential tails for the distribution of ξ and $\mathcal{F}$ does not imply exponential tails for the distribution of $\xi\mathcal{F}$, unless the respectively other random variable is almost surely bounded. To see why, consider for instance the case when $\mathcal{F}$ satisfies that there exists $\widetilde{c} > 0$ such that $\mathbb{E}[e^{\widetilde{c}\mathcal{F}}] = \infty$ and ξ has unbounded support. Observe that

$$\mathbb{E}\Big[e^{-c\xi\mathcal{F}}\Big] = \sum_{k=0}^{\infty} \mathbb{E}\Big[e^{-ck\mathcal{F}}\Big]\mathbb{P}(\xi=k).$$

Now, since ξ has unbounded support, for all $c > 0$ we can take $k > 0$ large enough such that $ck > \widetilde{c}$ and $\mathbb{P}(\xi = k) > 0$, which implies that

$$\mathbb{E}\Big[e^{-c\xi\mathcal{F}}\Big] \geqslant \mathbb{E}\Big[e^{-\widetilde{c}\mathcal{F}}\Big]\mathbb{P}(\xi=k) = \infty.$$

Theorema 2.2.2. *Consider the inhomogeneous contact process on the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. Suppose that only the root of the tree is initially infected. Assume that ξ and $\mathcal{F}$ are unbounded and one of the following two conditions holds*

$$\limsup_{f\to\infty} \frac{\log \mathbb{P}(\mathcal{F} > f)}{\log f} = -C_1 \quad \textit{for some} \quad C_1 \in [0,\infty), \tag{B}$$

$$\limsup_{k\to\infty} \frac{\log \mathbb{P}(\xi = k)}{k} = -C_2, \quad \textit{for some} \quad C_2 \in [0,\infty), \tag{C}$$

Then $\lambda_1 = \lambda_2 = 0$, i.e., the process survives strongly for any $\lambda > 0$.

Remark 2.2.2. The assumption that the fitness is lower bounded from below by 1, allows us, as mentioned earlier, to compare our model with the contact process with constant fitness. This, together with the monotonicity property is used at various places in the proofs. For instance, in our proof of Theorem 2.2.1, we use this assumption in order to control the expected survival times of the contact process, see Lemma

2.4.1 below. However, this is the only part in the proof where we use this assumption, so it seems plausible that we can improve this argument. On the other hand, in Theorem 2.2.2 the lower bound of the fitness helps us to study the persistence of the infection in a finite star using the monotonicity of the process and results already known in the literature for the standard contact process (see Lemma 2.5.4 below).

Remark 2.2.3. The behaviour of the contact process on the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ together with the offspring distribution satisfying Assumption (**C**) in Theorem 2.2.2 is similar but not exactly the same to the behaviour of the contact process with constant fitness, i.e. when $\mathcal{F} \equiv 1$. More precisely, as mentioned earlier, Huang and Durret [41] showed that the contact process with constant fitness and under the condition $\mathbb{E}[e^{c\xi}] = \infty$ for all $c > 0$ or equivalently

$$\limsup_{k\to\infty} \frac{\log \mathbb{P}(\xi = k)}{k} = 0,$$

exhibits only the strong survival phase (see [41, Theorem 1.4]). Roughly speaking, this condition tells us that the tree has high degree vertices in which the infection persists for long time, allowing the process to always survive.

Note that the influence of the fitness changes means that there are situations when the inhomogeneous contact is always supercritical, whereas the standard contact process on a Galton-Watson tree would exhibit a phase transition. Indeed this can be achieved in the following two ways for unbounded fitness and offspring distributions: either suppose the offspring distribution has exponential moments but the fitness satisfies Assumption (**B**) above or suppose the offspring distribution satisfies Assumption (**C**) for some $C_2 > 0$ and the fitness is unbounded. Then, in both cases the offspring distribution has exponential moments, so the standard contact process has a phase transition, while Theorem 2.2.2 guarantees that the inhomogeneous contact process is always supercritical.

Remark 2.2.4. Note that there are cases not captured by either of the two conditions in Theorem 2.2.1 and Theorem 2.2.2. For example, consider the following distributions for the offspring and the fitness:

$$\mathbb{P}(\xi = k) = \eta\, e^{-k^{\alpha}}, \qquad k \in \mathbb{N} \quad \text{and} \quad \alpha > 1,$$

$$\mathbb{P}(\mathcal{F} > f) = e^{-f^{\beta}}, \qquad f \geqslant 1 \quad \text{and} \quad \beta \in (0, 1],$$

where η is the normalizing constant. It is not difficult to see that, both distributions fulfill neither Assumption (**B**) nor (**C**) of Theorem 2.2.2. Moreover, the distribution of $\xi\mathcal{F}$ does not satisfy Assumption (**A**) of Theorem (2.2.1). In fact, for all $c > 0$ there exists $k > 0$ large enough such that

$$\mathbb{E}\Big[e^{c\,\xi\mathcal{F}}\Big] \geqslant \mathbb{E}\Big[e^{c\,k\mathcal{F}}\Big]\mathbb{P}(\xi = k) = \infty.$$

So far we have not been able to prove a general result about the phase transition when the distribution of $\xi\mathcal{F}$ satisfies

$$\mathbb{E}\Big[e^{c\,\xi\mathcal{F}}\Big] = \infty, \qquad \text{for all} \quad c > 0.$$

Having a result under the latter condition and combining it with Theorem 2.2.1, would give us the complete characterisation on the existence of a phase transition of the contact process on Galton-Watson trees with fitness. Nonetheless, this problem requires different techniques to those developed here and is currently on-going research.

Main ideas of proofs. In this section, we give a short overview of the proofs of the main theorems before giving the full proofs.

The proof of Theorem 2.2.1 is based on two main ideas which we adapt from [13], where it is used for the standard contact process. First, we use a recursive analysis on Galton-Watson trees that allows us to control the expected survival times. To this end, we consider the contact process on the finite tree $\mathcal{T}_L$ which corresponds to the restriction of $\mathcal{T}$ to the first L generations. The first goal is to show that, for small enough λ, the expected survival time of $\mathbf{CP}(\mathcal{T}_L; \mathbf{1}_\rho)$ is bounded from above uniformly in L. As in [13], we use a coupling, where we add an extra vertex only adjoined to the root that is always infected. In this way, the process on the subtrees rooted in the children of the root can by independence be compared to the full process on a tree (with extra root) restricted to $L-1$ vertices (see Lemma 2.4.1 below).

The second idea is to study the probability that the infection set of $\mathbf{CP}(\mathcal{T}_L; \mathbf{1}_\rho)$ goes deeper that a given height. More precisely, we prove that the probability that the infection travels deeper than a given heigh decays exponentially in the height (similarly as in [13]). The main strategy is to investigate the stationary distribution of another modification of the original contact process in finite Galton-Watson trees and relate it to the extinction time. Finally, we can establish our result by combining the two ideas.

The proof of Theorem 2.2.2 is based on techniques developed by Pemantle in [64]. He used these techniques to show an upper bound for the threshold value λ_2 for the contact process with constant fitness defined on Galton-Watson trees. Furthermore, when the offspring distribution is given by the following distribution $\mathbb{P}(\xi = k) = ce^{-k^\gamma}$ with $c > 0$ and $\gamma < 1$, he proved that $\lambda_2 = 0$. More recently, this strategy was extended by Durrett and Huang in [41, Theorem 1.4] in order to show that $\lambda_2 = 0$ if $\mathscr{L}(\xi)$ has subexponential tails. The same strategy holds in our case but we need to take extra care because of the presence of fitness.

First we estimate the survival time for the contact process with fitness on a finite star. For the case with constant fitness, Berger et al. in [9] showed that the infection can survive for time e^{cd} in a star of size d with positive probability for some $c > 0$. Now, in the case with fitness we prove that, if the root of the star is initially infected, it will keep a large number of infected leaves for a long time with probability very close to 1 if the fitness is large enough (see Lemma 2.5.4). With this in hand, we study the contact process in a star where a single path of a given length is added to some leaf of the star (see Lemma 2.5.6). The intuitive strategy for the proof of Theorem 2.2.2 is to push the infection to stars with high size and fitness in a suitable generation. As a next step we bring the infection back to the root appealing to Lemma 2.6.2 given by Pemantle in [64].

Overview of structure. The remaining chapter is structured as follows. In Section 2.3 the graphical representation for the contact process is introduced as well as some other useful properties. In particular, we will also see that started with a single infected vertex, the set of infected vertices stays finite at all times. Section 2.4 is devoted to the proof of Theorem 2.2.1. In Section 2.5, we prove preliminary results regarding to contact process with fitness on finite stars, needed for our purpose. Finally, in Section 2.6 the proof of Theorem 2.2.2 is completed.

2.3 Properties of the inhomogeneous contact process

In this section we provide an equivalent description of our model by a convenient *graphical representation* based on the construction given in [54, Chapter 1] for the case of constant fitness. We point out some properties which are direct consequences of the construction. Further, we show that under the assumption $\mathbb{E}[\xi] < \infty$, the contact process does not explode in finite time almost surely, so that the set of infected vertices at any time is finite almost surely.

Recall that $V(\mathcal{T})$ denotes the set of vertices in $\mathcal{T}$. Denote by $E(\mathcal{T}) = \{(u, v) : u, v \in V(\mathcal{T})\}$ the set of directed edges. Conditionally on the fitness values $\mathbb{F}(\mathcal{T})$, let $\{N_v\}_{v\in V(\mathcal{T})}$ be i.i.d. Poisson (point) processes with rate 1 and $\{N_{(v,u)}\}_{(v,u)\in E(\mathcal{T})}$ i.i.d. Poisson processes with rate $\lambda\mathcal{F}_v\mathcal{F}_u$. All the Poisson processes are mutually independent. For each $v \in V(\mathcal{T})$, a recovery symbol $*$ is placed at the point $(v,t) \in \mathcal{T} \times [0,\infty)$, at each event time t of N_v. For each $(v,u) \in E(\mathcal{T})$, an infection arrow $\longrightarrow$ is placed from (v,t) to (u,t), at each event time t of $N_{(v,u)}$. Then, given any initial condition, the contact process can be defined as follows: an infected vertex v infects another vertex u at a time t if t is in $N_{(v,u)}$. Similarly, an infected vertex recovers any time that is in N_v.

One advantage of the graphical construction is that it provides a joint coupling of the processes with different infection rules or different initial states. We state two useful facts about the contact process that we will use later in our proofs and that are consequence of using the graphical representation and the fact that we can e.g. easily couple Poisson process with different rates.

- *Monotonicity in infection rates.* Let $\mathbb{F}_1(\mathcal{T}) = (\mathcal{F}_v^1)_{v \in V(\mathcal{T})}$ and $\mathbb{F}_2(\mathcal{T}) = (\mathcal{F}_v^2)_{v \in V(\mathcal{T})}$ be sequences of fitness such that $\mathcal{F}_v^1 \leqslant \mathcal{F}_v^2$ a.s. for all $v \in V(\mathcal{T})$. Let $(X_t^1) \sim \mathbf{CP}((\mathcal{T}, \mathbb{F}_1(\mathcal{T})); \mathbf{1}_A)$ and $(X_t^2) \sim \mathbf{CP}((\mathcal{T}, \mathbb{F}_2(\mathcal{T})); \mathbf{1}_A)$, for any $A \subset V(\mathcal{T})$. Then, we can couple both processes, such that for any $t \geqslant 0$, we have $X_t^1 \leqslant X_t^2$, i.e.,

$$X_t^1(v) \leqslant X_t^2(v), \quad \text{for all} \quad v \in V(\mathcal{T}),$$

  (see e.g. [66, Section 1.2] for the case of the contact process with fitness in a completely deterministic graph).

- Consider $(X_t^1) \sim \mathbf{CP}(\mathcal{T}; \mathbf{1}_A)$, with any $A \subset V(\mathcal{T})$. Let $\mathbf{I}$ be any subset of $[0,\infty)$. Define (X_t^2) to be a coupled process of (X_t^1) that has the same initial state, infection and recoveries, except that the recoveries at a fixed vertex $v \in V(\mathcal{T})$ are ignored at times $t \in \mathbf{I}$. Then, we can couple both processes, such that for all $t \geqslant 0$, we have $X_t^1 \leqslant X_t^2$, i.e.,

$$X_t^1(v) \leqslant X_t^2(v), \quad \text{for all} \quad v \in V(\mathcal{T}),$$

  (see for instance [13, Lemma 2.2] for the case of the contact process with constant fitness).

The next result that we show is that if we start with a finite configuration, then almost surely the configuration remains finite for all times. Our argument adapts the proof of Durret [27, Theorem 2.1] to our setting with the additional difficulty that the underlying graph is random and we have unbounded rates.

Let r be an arbitrary non-negative integer. Let denote by V_r the set of vertices in generation r in the tree $\mathcal{T}$, i.e.,

$$V_r = \{v \in V(\mathcal{T}) : d(\rho, v) = r\}, \tag{2.1}$$

where $d(\cdot, \cdot)$ is the graph distance between two vertices in the tree. Let t_0 be a positive number. Define the graph G_{t_0}, a spanning subgraph of the Galton-Watson tree $\mathcal{T}$, by saying that two vertices u, v in $V(\mathcal{T})$ with $d(\rho, u) < d(\rho, v)$ that are neighbours in $\mathcal{T}$ are also neighbours in G_{t_0} if

$$N_{(u,v)}\Big([0, t_0]\Big) \geqslant 1.$$

Let denote by $\mathscr{C}_{t_0}(v)$ the vertex set of the connected component of v in the graph G_{t_0}. We will first show that if t_0 is small enough, then the component sizes in G_{t_0} are finite. Then, we will make the connection to the contact process as follows: If we start with only the root infected, then by the graphical construction and the fact that we are working on a tree, every vertex that has been infected by time t_0 in the contact process is contained in $\mathscr{C}_{t_0}(\rho)$.

Lemma 2.3.1. *Assume that $\mu = \mathbb{E}[\xi] < \infty$. If t_0 is small enough, then for any finite $\ell \in \mathbb{N}$, we have*

$$\mathbb{P}\Bigg(\Bigg|\bigcup_{v \in \bigcup_{r=0}^{\ell} V_r} \mathscr{C}_{t_0}(v)\Bigg| < \infty\Bigg) = 1.$$

Proof. We start by showing that $|\mathscr{C}_{t_0}(\rho)| < \infty$ almost surely. For this result, by Borel-Cantelli, it suffices to show that

$$\sum_{r=0}^{\infty} \mathbb{P}\Big(\mathscr{C}_{t_0}(\rho) \cap V_r \neq \emptyset\Big) < \infty.$$

We begin by denoting the set of non-intersecting paths of length r in the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ started at the root as follows

$$C_r = \{v = (v_0, \dots, v_r) : v_i \in V_i,\ i \in \{0, \dots, r\}\}.$$

Note that with this notation $v_0 = \rho$ and the number of paths of length r corresponds to the number of vertices in generation r, that is, $|C_r| = |V_r|$.

Let $c = (v_0, \dots, v_r) \in C_r$ be a path of length r in $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. For each $i \in \{0, \dots, r\}$, by definition of G_{t_0}, we have that the probability that the vertices v_{i-1} and v_i are connected in G_{t_0} is

$$1 - \exp\Big(-\lambda \mathcal{F}_{v_{i-1}} \mathcal{F}_{v_i} t_0\Big).$$

Thus, conditioning on $(\mathcal{T},\mathbb{F}(\mathcal{T}))$, the probability that there is a vertex in generation r that is also in $\mathscr{C}_{t_0}(\rho)$ is

$$\mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(\mathscr{C}_{t_0}(\rho)\cap V_r\neq\emptyset\Big)\leqslant |V_r|\max_{v\in C_r}\prod_{i=1}^{r}\Big(1-\exp\big(-\lambda\mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}t_0\big)\Big).$$

For an upper bound on the maximum in the right-hand side above, we have to show that with sufficiently high probability, each path has sufficiently many disjoint pairs of vertices when $\mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}$ is bounded by a large (but finite) constant. More precisely, let k be a positive constant, which we will choose at the end, and define the following event

$$E_r=\left\{\forall v=(v_0,\dots,v_r)\in C_r:\left|\left\{i=0,\dots,\left\lceil\frac{r}{2}\right\rceil-1:\ \mathcal{F}_{v_{2i}}\mathcal{F}_{v_{2i+1}}\leqslant k\right\}\right|\geqslant\frac{1}{2}\left\lceil\frac{r}{2}\right\rceil\right\},$$

where $\lceil x\rceil$ denotes the ceiling of the number x. On the event $E_r\cap\{|V_r|\leqslant(2\mu)^r\}$ and taking $t_0=t_0(\mu,\lambda,k)$ sufficiently small, we have

$$\mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(\mathscr{C}_{t_0}(\rho)\cap V_r\neq\emptyset\Big)\leqslant(2\mu)^r\max_{v\in C_r}\prod_{i=0}^{\lceil r/2\rceil-1}\left(1-e^{-\lambda kt_0}\right)\leqslant(2\mu)^r\left(1-e^{-\lambda kt_0}\right)^{\lceil r/2\rceil}\leqslant\left(\frac{1}{2}\right)^r.$$

Therefore, together with the previous upper bound we obtain

$$\begin{aligned}\mathbb{P}\Big(\mathscr{C}_{t_0}(\rho)\cap V_r\neq\emptyset\Big)&\leqslant\mathbb{P}\Big(\{\mathscr{C}_{t_0}(\rho)\cap V_r\neq\emptyset\}\cap E_r\cap\{|V_r|\leqslant(2\mu)^r\}\Big)+\mathbb{P}\Big(\{|V_r|\geqslant(2\mu)^r\}\Big)\\&\qquad+\mathbb{P}\Big(E_r^c\cap\{|V_r|\leqslant(2\mu)^r\}\Big)\\&\leqslant\left(\frac{1}{2}\right)^r+\mathbb{P}\Big(\{|V_r|\geqslant(2\mu)^r\}\Big)+\mathbb{P}\Big(E_r^c\cap\{|V_r|\leqslant(2\mu)^r\}\Big).\end{aligned}$$

What remains is to show that the second and third probability on the right-hand side are also summable. In order to control the number of vertices at level r, we can use Markov's inequality, to deduce that

$$\mathbb{P}\Big(|V_r|\geqslant(2\mu)^r\Big)\leqslant\frac{\mathbb{E}\big[|V_r|\big]}{(2\mu)^r}\leqslant\left(\frac{1}{2}\right)^r.$$

Furthermore, we bound

$$\mathbb{P}\Big(E_r^c\cap\{|V_r|\leqslant(2\mu)^r\}\Big)\leqslant(2\mu)^r\mathbb{P}\left(\left|\left\{i=0,\dots,\left\lceil\frac{r}{2}\right\rceil-1:\ \mathcal{F}_{2i}\mathcal{F}_{2i+1}\geqslant k\right\}\right|\geqslant\frac{1}{2}\left\lceil\frac{r}{2}\right\rceil\right),$$

where $\mathcal{F}_0,\dots,\mathcal{F}_r$ are i.i.d random variables with same distribution as $\mathcal{F}$. Moreover, we observe that the random variable

$$\left|\left\{i = 0, \dots, \left\lceil \frac{r}{2} \right\rceil - 1 :\ \mathcal{F}_{2i}\mathcal{F}_{2i+1} \geqslant k\right\}\right| = \sum_{i=0}^{\lceil r/2 \rceil - 1} \mathbf{1}_{\{\mathcal{F}_{2i}\mathcal{F}_{2i+1} \geqslant k\}}$$

has a binomial distribution $\mathbf{Bin}(\lceil r/2 \rceil, p_k)$ where $p_k = \mathbb{P}(\mathcal{F}_0\mathcal{F}_1 \geqslant k)$. Note that as $k \to \infty$, we have that $p_k \to 0$. By using a large deviation bound for binomial distributions, see e.g. [70, Corollary 2.20], for k large enough such that $p_k < \frac{1}{2}$, we obtain

$$\mathbb{P}\left(\sum_{i=0}^{\lceil r/2 \rceil - 1} \mathbf{1}_{\{\mathcal{F}_{2i}\mathcal{F}_{2i+1} \geqslant k\}} \geqslant \frac{1}{2}\left\lceil \frac{r}{2} \right\rceil\right) \leqslant \exp\left(-\left\lceil \frac{r}{2} \right\rceil I_{p_k}(1/2)\right),$$

where

$$I_{p_k}(1/2) = p_k - \frac{1}{2} - \frac{1}{2}\log\left(\frac{p_k}{1/2}\right).$$

In addition, since $p_k \to 0$ so that $I_{p_k}(1/2) \to \infty$, we can choose $k = k(\mu)$ arbitrary large such that $2\mu e^{-\frac{1}{2}I_{p_k}(1/2)} \leqslant 1/2$. Therefore, we obtain that

$$\mathbb{P}\Big(E_r^c \cap \{|V_r| \leqslant (2\mu)^r\}\Big) \leqslant (2\mu)^r \exp\left(-\frac{r}{2}I_{p_k}(1/2)\right) \leqslant \left(\frac{1}{2}\right)^r.$$

Putting the pieces together, we deduce that

$$\sum_{r=0}^{\infty} \mathbb{P}\Big(\mathscr{C}_{t_0}(\rho) \cap V_r \neq \emptyset\Big) \leqslant 3\sum_{r=0}^{\infty}\left(\frac{1}{2}\right)^r < \infty,$$

which implies by the Borel-Cantelli lemma, that almost surely, for r sufficiently large $\mathscr{C}_{t_0}(\rho) \cap V_r = \emptyset$ with probability 1, so that $|\mathscr{C}_{t_0}(\rho)|$ is finite almost surely.

The same argument shows that almost surely for each $v \in V_\ell$, we have that $\mathscr{C}_{t_0}(v)$ restricted to the vertices in the subtree of $\mathcal{T}$ rooted at v is finite. This immediately implies the statement of the lemma, as $\bigcup_{r=0}^{\ell} V_r$ is finite almost surely. $\square$

Theorema 2.3.2. *Assume that $\mu = \mathbb{E}[\xi] < \infty$. Consider $(X_t) \sim \mathbf{CP}(\mathcal{T}, \mathbf{1}_A)$ started with any finite set $A \subset V(\mathcal{T})$ infected, the inhomogeneous contact process on $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. Then*

$$\mathbb{P}\Big(|X_t| < \infty,\ \forall t \geqslant 0\Big) = 1.$$

Proof. Let t_0 be as given in Lemma 2.3.1. Since the initial configuration of the contact process is finite, we can find a sufficiently large and finite k such that $A \subset \bigcup_{r=0}^{k} V_r$, so that by the graphical construction, we have that $X_t \subset \cup_{v \in \cup_{r=0}^{k} V_r} \mathscr{C}_{t_0}(v)$ for all $t \in [0, t_0]$.

Then according to Lemma 2.3.1, we have that

$$\mathbb{P}\Big(|X_t| < \infty,\ \forall t \in [0, t_0]\Big) = 1.$$

Let us now define a spanning subgraph G_{2t_0} of $\mathcal{T}$ as follows: two neighbouring vertices u, v in $\mathcal{T}$ with $d(\rho, u) < d(\rho, v)$ are connected if

$$N_{(u,v)}\Big([t_0, 2t_0]\Big) \geqslant 1.$$

As before, let $\mathscr{C}_{t_0}(\rho)$ be the connected component of ρ in the graph G_{t_0}. For each $v \in \mathscr{C}_{t_0}(\rho)$, we denote by $\mathscr{C}_{2t_0}(v)$ the connected component of v in the graph G_{2t_0}. We observe that by the graphical construction

$$X_t \subset \bigcup_{v \in \mathscr{C}_{t_0}(\rho)} \mathscr{C}_{2t_0}(v), \qquad \text{for all} \quad t \in [t_0, 2t_0].$$

Then we can argue that by the independence of G_{t_0} and G_{2t_0} (due to the independence of the increments of the Poisson point processes), for any $k \geqslant 1$,

$$\begin{aligned}
\mathbb{P}\Bigg(\Bigg|\bigcup_{v \in \mathscr{C}_{t_0}(\rho)} \mathscr{C}_{2t_0}(v)\Bigg| < \infty\Bigg) &\geqslant \mathbb{P}\Bigg(\Bigg|\bigcup_{v \in \cup_{r=0}^{k} V_r} \mathscr{C}_{2t_0}(v)\Bigg| < \infty,\ |\mathscr{C}_{t_0}(\rho)| \leqslant k\Bigg) \\
&= \mathbb{E}\Bigg[\mathbb{P}_{\mathcal{T},\mathbb{F}}\Bigg(\Bigg|\bigcup_{v \in \cup_{r=0}^{k} V_r} \mathscr{C}_{2t_0}(v)\Bigg| < \infty\Bigg)\mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(|\mathscr{C}_{t_0}(\rho)| \leqslant k\Big)\Bigg] \\
&\geqslant \mathbb{P}\Bigg(\Bigg|\bigcup_{v \in \cup_{r=0}^{k} V_r} \mathscr{C}_{2t_0}(v)\Bigg| < \infty\Bigg) - \mathbb{P}\Big(|\mathscr{C}_{t_0}(\rho)| \geqslant k\Big) \\
&= 1 - \mathbb{P}\Big(|\mathscr{C}_{t_0}(\rho)| \geqslant k\Big).
\end{aligned}$$

Letting k tends to ∞ and appealing to Lemma 2.3.1 we deduce that the right-hand side converges to 1, which implies that

$$\mathbb{P}\Big(|X_t| < \infty,\ \forall t \in [t_0, 2t_0]\Big) = 1.$$

Iterating the argument we can conclude that $\mathbb{P}\Big(|X_t| < \infty,\ \forall t \in [nt_0, (n+1)t_0]\Big) = 1$ for all non-negative integers n and thus the proof is completed. $\square$

2.4 Proof of Theorem 2.2.1

This section is devoted to proving Theorem 2.2.1. The proof follows similar arguments as those used in Bhamidi et. al. [13, Theorem 1], although the presence of fitness leads to significant changes. The proof will consist of a series of lemmas. We begin by defining the contact process on a rooted tree with an extra parent added to the root. This modified process allows us to obtain our first result, Lemma 2.4.1, which states that for small enough λ the survival time of the original process in any tree restricted to the first L generations is bounded by a constant uniform in L. Afterwards, we define a suitable delayed version of the contact process. Using this delayed version, we shows that the probability that the infection in the original contact process travels deeper than a given height decays exponentially in the height, see Lemma 2.4.3 below. At the end of this section, we will combine the previous results to establish Theorem 2.2.1.

First we set up some extra notations for this section. Let D denote the degree of the root ρ and let $v_1, \ldots, v_D$ be the children of ρ with fitness $\mathcal{F}_{v_1}, \ldots, \mathcal{F}_{v_D}$. Let $\mathcal{T}_{v_1}, \ldots, \mathcal{T}_{v_D}$ be the subtrees rooted in $v_1, \ldots, v_D$ respectively. We denote by $\mathcal{T}_L$ the tree obtained by the restricting of the tree $\mathcal{T}$ to the first L generations. Denote by $\mathcal{T}_L^+$ the tree $\mathcal{T}_L$, but with an extra parent ρ^+ for the vertex ρ. We will assume that ρ^+ has constant fitness, i.e., $\mathcal{F}_{\rho^+} = 1$. Let $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ denote the contact process in the tree $\mathcal{T}_L^+$ with the extra root ρ^+. This process has the same infection and recovery rates as $\mathbf{CP}(\mathcal{T}_L; \mathbf{1}_\rho)$, but the extra parent ρ^+ is permanently infected and does not have a recovery clock attached to itself. Since the state of the root ρ^+ never changes, we will throughout specify the state of $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ by specifying the state restricted to $\mathcal{T}_L$, so that $\mathbf{0}$ denotes the state where the only infected vertex is ρ^+.

The first step to prove Theorem 2.2.1 is to show that for any L, the expected excursion time for the contact process over $\mathcal{T}_L$ is bounded from above uniformly in L if the parameter λ is small enough.

Lemma 2.4.1. *Suppose that ξ and $\mathcal{F}_\rho$ satisfy Assumption $(\mathbf{A})$. For $L \in \mathbb{N}$, let R_L be the first time when $\mathbf{CP}(\mathcal{T}_L; \mathbf{1}_\rho)$ reaches state $\mathbf{0}$. Then there exists a constant $\lambda_0 > 0$ such that for any $\lambda \leqslant \lambda_0$ and L,*

$$\mathbb{E}\Big[R_L\Big] \leqslant e^{1/2}\, \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}.$$

Proof. Let S_L be the first time when $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ reaches state $\mathbf{0}$ on $\mathcal{T}_L$. By the monotonicity of the contact process and since $\mathcal{F}_\rho \geqslant 1$ a.s., we observe that

$$\mathbb{E}\Big[R_L\Big] \leqslant \mathbb{E}\Big[\mathcal{F}_\rho R_L\Big] \leqslant \mathbb{E}\Big[\mathcal{F}_\rho S_L\Big].$$

Let us now introduce a modification of the contact process $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+;\mathbf{1}_\rho)$, where the recoveries at the root only occur when none of its descendants are infected. We denote the process by $(\widetilde{X}_t) \sim \widetilde{\mathbf{CP}}_{\rho^+}(\mathcal{T}_L^+;\mathbf{1}_\rho)$. Let $\widetilde{S}_L$ be the first time when $(\widetilde{X}_t)$ reaches the all-healthy state on $\mathcal{T}_L$. By ignoring the recovery attempts of the root when any vertices other than ρ and ρ' are infected, we can couple the processes such that $S_L \leqslant \widetilde{S}_L$. Hence, it suffices to establish the claimed upper bound for $\mathbb{E}[\mathcal{F}_\rho \widetilde{S}_L]$.

Let denote by $\mathcal{T}_{v_i}^+$ the tree rooted at v_i and can now treat ρ as the extra permanently infected parent of v_i. Define S_{L-1} and S_{L-1}^i to be the first time when $\mathbf{CP}_{\rho^+}(\mathcal{T}_{L-1}^+;\mathbf{1}_\rho)$ and $\mathbf{CP}_\rho(\mathcal{T}_{v_i}^+;\mathbf{1}_{v_i})$ reaches state $\mathbf{0}$, respectively. Note that $S_{L-1} \overset{(d)}{=} S_{L-1}^i$. We shall bound $\mathbb{E}[\mathcal{F}_\rho S_L]$ in terms of $\mathbb{E}[\mathcal{F}_\rho S_{L-1}]$. Define the following product chain

$$(X_t^{\otimes}) \sim \mathbf{CP}_\rho^{\otimes}\Big(\mathcal{T}_L;\mathbf{1}_{v_i}\Big) := \Big(\otimes_{j=1,j\neq i}^{D}\mathbf{CP}_\rho\Big(\mathcal{T}_{v_j}^+;\mathbf{0}\Big)\Big) \otimes \mathbf{CP}_\rho\Big(\mathcal{T}_{v_i}^+;\mathbf{1}_{v_i}\Big),$$

and denote by $\widetilde{S}_i^{\otimes}$ the first time when this product chain when restricted to $\cup_{j=1}^{D}\mathcal{T}_{v_j}$ reaches $\mathbf{0}$ started from $\mathbf{1}_{v_i}$. Further, define $\widetilde{S}^{\otimes}$ as the following average

$$\widetilde{S}^{\otimes} = \frac{1}{D}\sum_{i=1}^{D} \mathcal{F}_{v_i}\widetilde{S}_i^{\otimes}. \tag{2.2}$$

Now, given the tree $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$, we can control the expectation of $\widetilde{S}_L$ by the expectation of $\widetilde{S}^{\otimes}$ using the process $(\widetilde{X}_t) \sim \widetilde{\mathbf{CP}}_{\rho^+}(\mathcal{T}_L^+;\mathbf{1}_\rho)$. If we start with $\widetilde{X}_0 = \mathbf{1}_\rho$, then the next step in the chain is either that the root recovers or the root infects one of its children. Moreover, an excursion to $\mathbf{0}$ of $(\widetilde{X}_t)$ can be described as follows:

- Possibility 1. We have reached $\tilde{S}_L$ as soon as ρ recovers before infecting one of its children. Conditioning on $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$ this event happens with probability

  $$p := \frac{1}{1 + \lambda \mathcal{F}_\rho \sum_{j=1}^{D} \mathcal{F}_{v_j}}.$$

  Also p is the expected waiting time to see the recovery of ρ conditional on this event happening first.

- Possibility 2. The root ρ infects any of its children, let say v_i, before ρ recovers. The child v_i is selected with probability proportional to the fitness of the root's

children, i.e., with probability
$$\frac{\mathcal{F}_{v_i}}{\sum_{i=1}^{D}\mathcal{F}_{v_i}}.$$
The expected time for ρ infects v_i is p (conditional on this happening first). Since ρ stays infected until everyone is healthy, we then need to wait for an excursion to $\mathbf{0}$ of the product chain $(X_t^{\otimes}) \sim \mathbf{CP}_\rho^{\otimes}(\mathcal{T}_L;\mathbf{1}_{v_i})$. When this excursion finishes we are back to the same starting point and either follow possibility 1 or 2.

Note that we have reached $\tilde{S}_L$ when we finally follow possibility 1. Hence, by splitting according to the number of times we have to go through the different possibilities, we obtain

$$\begin{aligned}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_L\right] &= \sum_{k=0}^{\infty} p\,(1-p)^k\left\{(k+1)p + k\frac{1}{\sum_{i=1}^{D}\mathcal{F}_{v_i}}\sum_{j=1}^{D}\mathcal{F}_{v_j}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_j^{\otimes}\right]\right\}\\ &= \sum_{k=0}^{\infty} p\,(1-p)^k\left\{(k+1)p + k\frac{D}{\sum_{i=1}^{D}\mathcal{F}_{v_i}}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^{\otimes}\right]\right\},\end{aligned}$$

where in the last equality we use the definition of $\widetilde{S}^{\otimes}$ given in (2.2). This series can be computed explicitly. Indeed, we have

$$\begin{aligned}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_L\right] &= p^2\sum_{k=0}^{\infty}(1-p)^k(k+1) + p\frac{D}{\sum_{i=1}^{D}\mathcal{F}_{v_i}}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^{\otimes}\right]\sum_{k=0}^{\infty}(1-p)^k k\\ &= 1 + p\frac{D}{\sum_{i=1}^{D}\mathcal{F}_{v_i}}\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^{\otimes}\right]\frac{(1-p)}{p^2} = 1 + \lambda D\mathcal{F}_\rho\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^{\otimes}\right].\end{aligned}$$

Then, by tower property of the conditional expectation we deduce

$$\mathbb{E}\left[\widetilde{S}_L|D,\mathcal{F}_\rho\right] = 1 + \lambda D\mathcal{F}_\rho\mathbb{E}\left[\widetilde{S}^{\otimes}|D,\mathcal{F}_\rho\right]. \tag{2.3}$$

On the other hand, we will find an estimate for the term on the right-hand side above by relating it to the stationary distributions of the product chain defined before. Let $\pi^{(D)}$ be the stationary distribution of the product chain $\mathbf{CP}_\rho^{\otimes}(\mathcal{T}_L)$ and π_i the stationary distribution of $\mathbf{CP}_\rho(\mathcal{T}_{v_i}^+)$. These stationary distributions are related in the following way
$$\pi^{(D)} = \otimes_{i=1}^{D}\pi_i. \tag{2.4}$$

Note that, for each fixed $i \in \{1,\dots,D\}$, the rate of leaving the state $\mathbf{0}$, and the expected return time to $\mathbf{0}$ of the chain $\mathbf{CP}_\rho(\mathcal{T}_{v_i}^+)$ are given by

$$q_i(\mathbf{0}) := \lambda\mathcal{F}_\rho\mathcal{F}_{v_i} \qquad \text{and} \qquad m_i(\mathbf{0}) := (\lambda\mathcal{F}_\rho\mathcal{F}_{v_i})^{-1} + \mathbb{E}_{\mathcal{T}_{v_i},\mathbb{F}}\left[S_{L-1}^i\right],$$

respectively. The stationary distribution $\pi_i(\mathbf{0})$ corresponds to the fraction of time that $\mathbf{CP}_\rho(\mathcal{T}_{v_i}^+)$ is at state $\mathbf{0}$. It can be calculated as the reciprocal of the product between the rate of leaving the state $\mathbf{0}$, and the expected return time to $\mathbf{0}$ (see for example [56, Theorem 3.5.3]). In other words, for each $i = 1, \ldots, D$, we deduce that

$$\pi_i(\mathbf{0}) = \frac{1}{1 + \lambda \mathcal{F}_\rho \mathcal{F}_{v_i} \mathbb{E}_{\mathcal{T}_{v_i}, \mathbb{F}} \left[S_{L-1}^i \right]}.$$

Similarly, the stationary distribution $\pi^{(D)}(\mathbf{0})$ corresponds to the fraction of time that $\mathbf{CP}_\rho^{\otimes}(\mathcal{T}_L)$ is at state $\mathbf{0}$ and it is given by $(q_D(\mathbf{0}) m_D(\mathbf{0}))^{-1}$, where

$$q_D(\mathbf{0}) := \lambda \mathcal{F}_\rho \sum_{i=1}^{D} \mathcal{F}_i \qquad \text{and} \qquad m_D(\mathbf{0}) = \frac{1}{q_D(\mathbf{0})} + \sum_{i=1}^{D} \frac{\lambda \mathcal{F}_\rho \mathcal{F}_{v_i}}{q_D(\mathbf{0})} \mathbb{E}_{\mathcal{T}_L, \mathbb{F}} \left[\widetilde{S}_i^{\otimes} \right],$$

and correspond to the rate of leaving $\mathbf{0}$ and the expected return time to $\mathbf{0}$ of $\mathbf{CP}_\rho^{\otimes}(\mathcal{T}_L)$, respectively. Hence, the reasoning above and the definition of $\widetilde{S}^{\otimes}$ show that

$$\pi^{(D)}(\mathbf{0}) = \frac{1}{1 + \lambda D \mathcal{F}_\rho \mathbb{E}_{\mathcal{T}_L, \mathbb{F}} \left[\widetilde{S}^{\otimes} \right]}.$$

Putting the pieces together and using (2.4), we deduce that

$$1 + \lambda D \mathcal{F}_\rho \mathbb{E}_{\mathcal{T}_L, \mathbb{F}} \left[\widetilde{S}^{\otimes} \right] = \prod_{i=1}^{D} \left(1 + \lambda \mathcal{F}_\rho \mathcal{F}_{v_i} \mathbb{E}_{\mathcal{T}_{v_i}, \mathbb{F}} \left[S_{L-1}^i \right] \right).$$

Then taking expectation conditionally on D and $\mathcal{F}_\rho$, we have

$$\begin{aligned} 1 + \lambda D \mathcal{F}_\rho \mathbb{E} \left[\widetilde{S}^{\otimes} | D, \mathcal{F}_\rho \right] &= \mathbb{E} \left[\prod_{i=1}^{D} \left(1 + \lambda \mathcal{F}_\rho \mathbb{E} \left[\mathcal{F}_{v_i} S_{L-1}^i | \mathcal{T}_{v_i}, \mathbb{F}(\mathcal{T}_{v_i}) \right] \right) \middle| D, \mathcal{F}_\rho \right] \\ &= \prod_{i=1}^{D} \left(1 + \lambda \mathcal{F}_\rho \mathbb{E} \left[\mathcal{F}_{v_i} S_{L-1}^i \right] \right). \end{aligned}$$

Plugging this back into (2.3), we get

$$\mathbb{E} \left[\widetilde{S}_L | D, \mathcal{F}_\rho \right] = \left(1 + \lambda \mathcal{F}_\rho \mathbb{E} \left[\mathcal{F}_\rho S_{L-1} \right] \right)^D \leqslant \exp \left(\lambda D \mathcal{F}_\rho \mathbb{E} \left[\mathcal{F}_\rho S_{L-1} \right] \right).$$

Then taking expectation over D and $\mathcal{F}_\rho$, and applying the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\mathbb{E}\Big[\mathcal{F}_\rho S_L\Big] \leqslant \mathbb{E}\Big[\mathcal{F}_\rho \widetilde{S}_L\Big] &\leqslant \mathbb{E}\Big[\mathcal{F}_\rho \exp\Big(\lambda D\mathcal{F}_\rho \mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]\Big)\Big] \\ &\leqslant \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho \mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]\Big)\Big]^{1/2}.\end{aligned} \tag{2.5}$$

For the last part of the proof we use an inductive argument over L appealing to our assumption that $D\mathcal{F}_\rho$ has exponential tails, i.e., $\mathbb{E}[e^{c\xi\mathcal{F}_\rho}] = M < \infty$ for some $c, M > 0$. We know that the latter assumption implies that $\mathcal{F}_\rho$ has finite moments, thus we can define finite constants K and λ_0 by

$$K = \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2} e^{1/2}\cdot \max\{\log M, 1\}, \qquad \lambda_0 = \frac{c}{2K}.$$

For the base case in the induction, observe that the random variable S_0 corresponds to the first time when $\mathbf{CP}_{\rho^+}(\mathcal{T}_0^+; \mathbf{1}_\rho)$ reaches state $\mathbf{0}$ on $\mathcal{T}_0$, in other words, when the recovery clock attached to ρ rings. Therefore, S_0 is exponentially distributed with parameter 1. Hence, since S_0 does not depend on $\mathcal{F}_\rho$, we get

$$\mathbb{E}\Big[\mathcal{F}_\rho S_0\Big] = \mathbb{E}[\mathcal{F}_\rho]\,\mathbb{E}[S_0] = \mathbb{E}[\mathcal{F}_\rho] \leqslant \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2} e^{1/2}.$$

To prove the inductive step, we assume $\mathbb{E}[\mathcal{F}_\rho S_{L-1}] \leqslant e^{1/2}\mathbb{E}[\mathcal{F}_\rho^2]^{1/2}$ holds. First, note that for any $\lambda \leqslant \lambda_0$, the definition of the constant K implies that

$$\begin{aligned}\frac{2\lambda\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}\Big]}{c} \leqslant \frac{\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}\Big]}{K} &\leqslant \frac{\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2} e^{1/2}}{\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2} e^{1/2}\cdot\max\{\log M, 1\}} \\ &= \frac{1}{\max\{\log M, 1\}} \leqslant 1.\end{aligned}$$

Thus, $g(x) = x^{\frac{2\lambda}{c}\mathbb{E}[\mathcal{F}_\rho S_{L-1}]}$ is a concave function. By Jensen's inequality and the exponential tails of $D\mathcal{F}_\rho$, we obtain for any $\lambda \leqslant \lambda_0$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]\Big)\Big] &= \mathbb{E}\left[\exp\left(cD\mathcal{F}_\rho\frac{2\lambda\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]}{c}\right)\right] \\ &\leqslant \exp\left\{\log M\frac{2\lambda\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]}{c}\right\}.\end{aligned}$$

From the inductive hypothesis and the definition of K, we have for all $\lambda \leqslant \lambda_0$,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho \mathbb{E}\left[\mathcal{F}_\rho S_{L-1}\right]\Big)\Big] &\leqslant \exp\left\{\log M \frac{2\lambda e^{1/2}\mathbb{E}[\mathcal{F}_\rho^2]^{1/2}}{c}\right\} \\ &\leqslant \exp\left\{\log M \frac{1}{\max\{\log M, 1\}}\right\} \leqslant e.\end{aligned}$$

Therefore, plugging this back into (2.5), we deduce the following inequality which holds for all $\lambda \leqslant \lambda_0$ and L,

$$\mathbb{E}\Big[\mathcal{F}_\rho S_L\Big] \leqslant \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho \mathbb{E}\left[F_\rho S_{L-1}\right]\Big)\Big]^{1/2} \leqslant e^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}.$$

This concludes the proof. □

The following result is a preparatory lemma describing the stationary mass of the empty state of a suitably defined delayed version of the contact process and the reader will see its importance later. Following [13] we introduce the delayed contact process. Let $\mathbf{T}_\phi$ denote a finite tree rooted at a vertex ϕ and set $\mathbf{T} = \mathbf{T}_\phi \setminus \{\phi\}$. For $x \in \{0,1\}^{V(\mathbf{T})}$ define the depth of x in $\mathbf{T}_\phi$ by

$$r\Big(x; \mathbf{T}_\phi\Big) = \max\{d(\phi, u) : x(u) = 1\},$$

where $d(\cdot,\cdot)$ denotes the graph distance between two vertices in the tree and where we define $r(\mathbf{0}; \mathbf{T}) = 0$.

Further, let us denote by Q_{xy} the transition rate from state x to state y of $\mathbf{CP}_\phi(\mathbf{T}_\phi)$. For a fixed $\theta \in (0,1)$, the delayed contact process $\mathbf{DP}_\phi(\mathbf{T}_\phi)$ is a continuous-time Markov chain on $\{0,1\}^{V(\mathbf{T})}$ with transition rates given by

$$Q^{(\theta)}_{xy} = \theta^{r\left(x;\ \mathbf{T}_\phi\right)} Q_{xy}. \tag{2.6}$$

Note that the process $\mathbf{DP}_\phi(\mathbf{T}_\phi)$ takes longer to change from state x to state y than $\mathbf{CP}_\phi(\mathbf{T}_\phi)$. Moreover, the delayed contact process spends more time in the states with greater depth. The following result use similar ideas to those used in the proof of [13, Proposition 3.6], but adapted to our setting.

Lemma 2.4.2. *Suppose that ξ and $\mathcal{F}_\rho$ satisfy Assumption $(\mathbf{A})$. Let L be an arbitrary non-negative integer. Let $\nu^\theta_{\mathcal{T}_L}$ denote the stationary distribution of $\mathbf{DP}_{\rho^+}(\mathcal{T}_L^+)$. Then there exist $K, \lambda_0 > 0$ such that $K\lambda_0 < 1$ and for all $\lambda \leqslant \lambda_0$ and L,*

$$\mathbb{E}\Big[\nu^\theta_{\mathcal{T}_L}(\mathbf{0})^{-1}\Big] \leqslant 2,$$

where $\theta \in (0,1)$ *is given by* $\theta = K\lambda$.

Recall that $v_1, \dots, v_D$ denote the children of the root ρ and $\mathcal{T}_{v_1}, \dots, \mathcal{T}_{v_D}$ the subtrees rooted in these children.

Proof. Consider $(X_t) \sim \mathbf{DP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ the delayed contact process. We denote by $(\widetilde{X}_t) \sim \widetilde{\mathbf{DP}}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ the following modification of the previous process. The process $(\widetilde{X}_t)$ shares the same infection and recovery clocks as (X_t) except in the root ρ. A recovery attempt at ρ at time t is only valid if $\widetilde{X}_t = \mathbf{1}_\rho$, so that there no other infected vertices apart from ρ and ρ^+. Let S_L^θ and $\widetilde{S}_L^\theta$ be the first excursion time when the delayed process $\mathbf{DP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ and the modified delayed process $\widetilde{\mathbf{DP}}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ reaches state $\mathbf{0}$, respectively. Further, let $\widetilde{S}_i^\theta$ denote the first time when $\mathbf{DP}_\rho(\mathcal{T}_L; \mathbf{1}_{v_i})$ becomes $\mathbf{0}$, and $\widetilde{S}^\theta$ is the following average

$$\widetilde{S}^\theta = \frac{1}{D} \sum_{i=1}^D \mathcal{F}_{v_i} \widetilde{S}_i^\theta. \tag{2.7}$$

An excursion to $\mathbf{0}$ of $(\widetilde{X}_t) \sim \widetilde{\mathbf{DP}}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ started from the initial configuration $\widetilde{X}_0 = \mathbf{1}_\rho$ can be described as follows:

- Possibility 1. We terminate if ρ recovers before infecting any of its children. Recall that $r(\mathbf{1}_\rho, \mathcal{T}_L^+) = 1$. From the definition of the transition rates in the delayed contact process, the probability that ρ recovers before infecting any of its children conditionally on $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$ is given by

  $$p := \frac{\theta}{\theta + \theta\lambda\mathcal{F}_\rho \sum_{j=1}^D \mathcal{F}_{v_j}} = \frac{1}{1 + \lambda\mathcal{F}_\rho \sum_{j=1}^D \mathcal{F}_{v_j}}.$$

  Furthermore, the expected waiting time to see the recovery of ρ is

  $$\frac{1}{\theta + \theta\lambda\mathcal{F}_\rho \sum_{j=1}^D \mathcal{F}_{v_j}} = \theta^{-1} p,$$

  conditionally on the fitness and on the event that the recovery happens first.

- Possibility 2. The root ρ infects any of its children, say v_i, before the recovery of ρ happens. Conditioning on the fitness, the vertex v_i is selected with probability proportional to the fitness of the root's children, i.e., with probability

  $$\frac{\mathcal{F}_{v_i}}{\sum_{j=1}^D \mathcal{F}_{v_j}}.$$

Since in the modified delayed contact process $(\widetilde{X}_t)$, the root ρ stays infected until everyone is healthy, we need to wait for an excursion to $\mathbf{0}$ in the subtree $\mathcal{T}_{v_i}$. The expected waiting time to this excursion is $\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}[\widetilde{S}_i^\theta]$, conditionally on vertex v_i being infected first. When this excursion finishes, we come back to the initial state and follow either possibility 1 or 2.

Note that we have reached state $\mathbf{0}$ as soon as the process follows possibility 1. Hence, by splitting according to the number of times the process follows possibility 2 before finally taking possibility 1, we obtain the expected excursion time to $\mathbf{0}$ of $(\widetilde{X}_t)$, given the tree $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$, as

$$\begin{aligned}
\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_L^\theta\right] &= \sum_{k=0}^{\infty} p\,(1-p)^k \left[\frac{1}{\theta}(k+1)p + \frac{k}{\sum_{i=1}^D \mathcal{F}_{v_i}} \sum_{j=1}^D \mathcal{F}_{v_j} \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_j^\theta\right]\right] \\
&= \sum_{k=0}^{\infty} p\,(1-p)^k \left[\frac{1}{\theta}(k+1)p + \frac{kD}{\sum_{i=1}^D \mathcal{F}_{v_i}} \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta\right]\right],
\end{aligned}$$

where in the last equality we use the definition of $\widetilde{S}^\theta$ given in (2.7). Similarly as in Lemma 2.4.1, we can explicitly calculate the previous series and obtain

$$\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_L^\theta\right] = \frac{1}{\theta}\left(1 + \theta\lambda D \mathcal{F}_\rho \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta\right]\right) \leqslant \frac{1}{\theta}\left(1 + \lambda D \mathcal{F}_\rho \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta\right]\right), \tag{2.8}$$

where in the last inequality we use that $\theta < 1$.

Now, we shall find an upper bound for $\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}[\widetilde{S}_L^\theta]$ in terms of $\mathbb{E}_{\mathcal{T}_{v_i},\mathbb{F}}[S_{L-1}^\theta]$. In order to do this, we will establish a relationship between $\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}[\widetilde{S}^\theta]$ and the stationary distribution of the delayed contact process. Denote by $S_{L-1}^{\theta,i}$ the first time when $\mathbf{DP}_\rho(\mathcal{T}_{v_i}^+; \mathbf{1}_{v_i})$ becomes $\mathbf{0}$. For $\mathbf{DP}_\rho(\mathcal{T}_{v_i}^+)$, the rate of leaving the state $\mathbf{0}$ and the expected return time to $\mathbf{0}$ are given by

$$q_i^\theta(\mathbf{0}) := \theta^{r\left(\mathbf{0};\mathcal{T}_{v_i}^+\right)} \lambda \mathcal{F}_\rho \mathcal{F}_{v_i} = \lambda \mathcal{F}_\rho \mathcal{F}_{v_i} \qquad \text{and} \qquad m_i^\theta(\mathbf{0}) := \left(q_i^\theta(\mathbf{0})\right)^{-1} + \mathbb{E}_{\mathcal{T}_{v_i},\mathbb{F}}\left[S_{L-1}^{\theta,i}\right].$$

Similarly, for the process $\mathbf{DP}_\rho(\mathcal{T}_L)$ we also find these two quantities

$$q^\theta(\mathbf{0}) := \theta^{r\left(\mathbf{0};\mathcal{T}_L^+\right)} \lambda \mathcal{F}_\rho \sum_{i=1}^D \mathcal{F}_{v_i} = \lambda \mathcal{F}_\rho \sum_{i=1}^D \mathcal{F}_{v_i},$$

and

$$m^\theta(\mathbf{0}) := \frac{1}{q^\theta(\mathbf{0})} + \sum_{i=1}^D \frac{\lambda \mathcal{F}_\rho \mathcal{F}_{v_i}}{q^\theta(\mathbf{0})} \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}_i^\theta\right],$$

where we recall that $\widetilde{S}_i^\theta$ is the first time when the modified process $\widetilde{\mathbf{DP}}_\rho(\mathcal{T}_L;\mathbf{1}_{v_i})$ reaches the completely recovered state $\mathbf{0}$. Let $\widetilde{\nu}^\theta_{\mathcal{T}_L}$ and $\nu^\theta_{\mathcal{T}_{v_i}}$ be the stationary distribution of $\mathbf{DP}_\rho(\mathcal{T}_L)$ and $\mathbf{DP}_\rho(\mathcal{T}_{v_i}^+)$, respectively. Also, we define $\nu^\otimes_{\mathcal{T}_L} := \otimes_{j=1}^D \nu^\theta_{\mathcal{T}_{v_j}}$. Therefore, we have

$$\widetilde{\nu}^\theta_{\mathcal{T}_L}(\mathbf{0}) = \frac{1}{1+\lambda D\mathcal{F}_\rho\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta\right]}, \quad \nu^\otimes_{\mathcal{T}_L}(\mathbf{0}) = \prod_{i=1}^D \frac{1}{1+\lambda\mathcal{F}_\rho\mathcal{F}_{v_i}\mathbb{E}_{\mathcal{T}_{v_i},\mathbb{F}}\left[S^{\theta,i}_{L-1}\right]}. \tag{2.9}$$

We can write each state $x \in \Omega_L := \{0,1\}^{V(\cup_{i=1}^D \mathcal{T}_{v_i})}$ of $\mathbf{DP}_\rho(\mathcal{T}_L)$ as $x = (x_i)_{i=1}^D$, where $x_i \in \Omega_i := \{0,1\}^{V(\mathcal{T}_{v_i})}$. Let $\pi_{\mathcal{T}_{v_i}}$ the stationary distribution $\mathbf{CP}_\rho(\mathcal{T}_{v_i}^+)$ and $\pi^\otimes_{\mathcal{T}_L} := \otimes_{i=1}^D \pi_{\mathcal{T}_{v_i}}$. Thanks to the relation (2.6) we can write the stationary distributions $\widetilde{\nu}^\theta_{\mathcal{T}_L}(x)$ and $\nu^\otimes_{\mathcal{T}_L}(x)$ in terms of $\pi^\otimes_{\mathcal{T}_L}$ and $\pi_{\mathcal{T}_{u_i}}$, respectively. More precisely,

$$\widetilde{\nu}^\theta_{\mathcal{T}_L}(x) = \frac{\theta^{-r(x;\mathcal{T}_L)}\pi^\otimes_{\mathcal{T}_L}(x)}{\sum_{y\in\Omega_L}\theta^{-r(y;\mathcal{T}_L)}\pi^\otimes_{\mathcal{T}_L}(y)} = \frac{\theta^{-r(x;\mathcal{T}_L)}\prod_{i=1}^D \pi_{\mathcal{T}_{v_i}}(x_i)}{\sum_{y\in\Omega_{\mathcal{T}_L}}\theta^{-r(y;\mathcal{T}_L)}\prod_{i=1}^D \pi_{\mathcal{T}_{v_i}}(y_i)},$$

and similarly

$$\nu^\otimes_{\mathcal{T}_L}(x) = \prod_{i=1}^D\left[\frac{\theta^{-r(x_i;\mathcal{T}_{v_i})}\pi_{\mathcal{T}_{v_i}}(x_i)}{\sum_{y_i\in\Omega_i}\theta^{-r(y_i;\mathcal{T}_{v_i})}\pi_{\mathcal{T}_{v_i}}(y_i)}\right] = \frac{\theta^{-\sum_{i=1}^D r(x_i;\mathcal{T}_{v_i})}\prod_{i=1}^D \pi_{\mathcal{T}_{v_i}}(x_i)}{\sum_{y\in\Omega_L}\theta^{-\sum_{i=1}^D r(y_i;\mathcal{T}_{v_i})}\prod_{i=1}^D \pi_{\mathcal{T}_{v_i}}(y_i)}.$$

Furthermore, for $x = (x_i)_{i=1}^D \in \Omega_L$ with $x_i \in \Omega_i$, we deduce that

$$r\Big(x;\mathcal{T}_L\Big) = \max\Big\{r\Big(x_i;\mathcal{T}_{v_i}\Big) : i=1,\dots,D\Big\} \leqslant \sum_{i=1}^D r\Big(x_i;\mathcal{T}_{v_i}\Big),$$

which implies $\nu^\otimes_{\mathcal{T}_L}(\mathbf{0}) \leqslant \widetilde{\nu}^\theta_{\mathcal{T}_L}(\mathbf{0})$. Using this fact together with (2.9), and plugging back into (2.8), we obtain that

$$\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta_L\right] \leqslant \frac{1}{\theta}\left(1+\lambda D\mathcal{F}_\rho\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\left[\widetilde{S}^\theta\right]\right) \leqslant \frac{1}{\theta}\prod_{i=1}^D\left(1+\lambda\mathcal{F}_\rho\mathcal{F}_{v_i}\mathbb{E}_{\mathcal{T}_{v_i},\mathbb{F}}\left[S^{\theta,i}_{L-1}\right]\right).$$

Taking expectations conditionally only on D and $\mathcal{F}_\rho$, we get

$$\mathbb{E}\left[\widetilde{S}^\theta_L|D,\mathcal{F}_\rho\right] \quad \leqslant \quad \frac{1}{\theta}\prod_{i=1}^D\left(1+\lambda\mathcal{F}_\rho\mathbb{E}\Big[\mathcal{F}_{v_i}S^{\theta,i}_{L-1}\Big]\right).$$

Moreover, taking account the monotonicity of the contact process, we have $S^\theta_L \leqslant \widetilde{S}^\theta_L$, which implies

$$\begin{aligned}\mathbb{E}\Big[S_L^\theta|D,\mathcal{F}_\rho\Big] &\leqslant \frac{1}{\theta}\prod_{i=1}^{D}\Big(1+\lambda\mathcal{F}_\rho\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}^\theta\Big]\Big) \leqslant \frac{1}{\theta}\left(1+\lambda\mathcal{F}_\rho\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}^\theta\right]\right)^D\\ &\leqslant \frac{1}{\theta}\exp\Big(\lambda D\mathcal{F}_\rho\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}^\theta\Big]\Big).\end{aligned}$$

Thus taking expectations over D and $\mathcal{F}_\rho$ and appealing to the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathbb{E}\Big[\mathcal{F}_\rho S_L^\theta\Big] &\leqslant \frac{1}{\theta}\mathbb{E}\left[\mathcal{F}_\rho\exp\Big(\lambda D\mathcal{F}_\rho\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}^\theta\Big]\Big)\right]\\ &\leqslant \frac{1}{\theta}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}^\theta\right]\Big)\Big]^{1/2}. \end{aligned}\tag{2.10}$$

Now, we shall apply an inductive argument over L to bound $\mathbb{E}\left[\mathcal{F}_\rho S_L^\theta\right]$. We assume that $\mathbb{E}\Big[e^{c\xi\mathcal{F}}\Big]=M$ for some $c,M>0$, and define the following constants

$$K=\max\left\{\frac{2\cdot 2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\log M}{c\log 2},\frac{2\cdot 2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{c},2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\right\},\quad \lambda_0=\frac{1}{2K},$$

and for any $\lambda\leqslant\lambda_0$ we set $\theta=\lambda K$. In particular note that choosing these constants guarantees that $\theta<1$. For the base case $L=0$, note that S_0^θ is an exponential random variable with parameter θ that is independent of $\mathcal{F}_\rho$. Therefore,

$$\mathbb{E}\Big[\mathcal{F}_\rho S_0^\theta\Big]=\mathbb{E}\Big[\mathcal{F}_\rho\Big]\mathbb{E}\Big[S_0^\theta\Big]\leqslant\frac{1}{\theta}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\leqslant\frac{1}{\theta}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}\leqslant\frac{1}{\theta}2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2},$$

so that the base case holds. Next, we assume $\mathbb{E}[\mathcal{F}_\rho S_{L-1}^\theta]\leqslant\theta^{-1}2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}$. Then we have

$$\frac{2\lambda\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}^\theta\right]}{c}\leqslant\frac{2\cdot 2^{1/2}\lambda\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{c\lambda K}\leqslant\frac{2\cdot 2^{1/2}\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{cK}\leqslant 1.$$

Hence, $g(x)=x^{\frac{2\lambda}{c}\mathbb{E}[\mathcal{F}_\rho^\theta S_{L-1}]}$ is a concave function, which permits us to apply Jensen's inequality to deduce

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(2\lambda D\mathcal{F}_\rho\mathbb{E}\Big[\mathcal{F}_\rho S_{L-1}^\theta\Big]\Big)\Big] &= \mathbb{E}\left[\exp\left(cD\mathcal{F}_\rho\frac{2\lambda\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}^\theta\right]}{c}\right)\right]\\ &\leqslant \exp\left(\log M\frac{2\lambda\mathbb{E}\left[\mathcal{F}_\rho S_{L-1}^\theta\right]}{c}\right)=M^{\frac{2\cdot 2^{1/2}\mathbb{E}[\mathcal{F}_\rho^2]^{1/2}}{cK}}\leqslant 2,\end{aligned}$$

where the last inequality is due to the definition of K. Plugging this back into (2.10), for any L and all $\lambda \leqslant \lambda_0$ with $\theta = K\lambda$, we have

$$\mathbb{E}\Big[\mathcal{F}_\rho S_L^\theta\Big] \leqslant \frac{1}{\theta} 2^{1/2} \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}. \tag{2.11}$$

Finally, $\nu_{\mathcal{T}_L}^\theta(\mathbf{0})$ the stationary distribution of $\mathbf{DP}_{\rho^+}(\mathcal{T}_L^+)$ in state $\mathbf{0}$, is $(q(\mathbf{0})m(\mathbf{0}))^{-1}$, where

$$q(\mathbf{0}) = r^{(\mathbf{0},\mathcal{T}_L^+)} \lambda \mathcal{F}_\rho \mathcal{F}_{\rho^+} = \lambda \mathcal{F}_\rho \qquad \text{and} \qquad m(\mathbf{0}) = \frac{1}{q(\mathbf{0})} + \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[S_L^\theta\Big],$$

and recall that S_L^θ is the first time when $(X_t) \sim \mathbf{DP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ reaches $\mathbf{0}$. Thus,

$$\nu_{\mathcal{T}_L}^\theta(\mathbf{0}) = \frac{1}{1 + \lambda \mathcal{F}_\rho \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[S_L^\theta\Big]} = \frac{1}{1 + \lambda \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[\mathcal{F}_\rho S_L^\theta\Big]}.$$

Taking expectation over the tree and the fitness in the reciprocal of the last expression and using (2.11), we we deduce the following inequality which holds for all $\lambda \leqslant \lambda_0$ and L,

$$\mathbb{E}\Big[\nu_{\mathcal{T}_L}^\theta(\mathbf{0})^{-1}\Big] = 1 + \lambda \mathbb{E}\Big[\mathcal{F}_\rho S_L^\theta\Big] \leqslant 1 + \frac{\lambda 2^{1/2} \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{\theta} \leqslant 1 + \frac{2^{1/2} \mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{K} \leqslant 2.$$

This concludes the proof. □

Recall that S_L was defined in Lemma 2.4.1 as the first time when $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+; \mathbf{1}_\rho)$ reaches the state $\mathbf{0}$ on $\mathcal{T}_L$. Let

$$H = \max\Big\{r\Big(X_t, \mathcal{T}_L^+\Big) : t \in [0, S_L]\Big\}$$

be the maximal depth that the contact process reaches until the time S_L. In the next lemma, we study the probability that the contact process in a given tree $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$ travels deeper than a given height in the tree. Essentially the proof mimics the steps of [13, Theorem 3.4]. However, we present it here for the sake of completeness.

Lemma 2.4.3. *Suppose that ξ and $\mathcal{F}_\rho$ satisfy Assumption* (**A**). *Let L be a non-negative integer. There exist constants $K, \lambda_0 > 0$ such for all $\lambda \leqslant \lambda_0$, $h > 0$ and $m > 0$, we have*

$$\mathbb{P}_{\mathcal{T}_L,\mathbb{F}}\Big(H > h\Big) \leqslant 2m(K\lambda)^h$$

with probability at least $1 - m^{-1}$ over the choice of $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$.

Proof. Let L be a non-negative integer and assume $h > 0$ and $m > 0$. We define the constants K, λ_0 and θ as were defined in Lemma 2.4.2. We denote by $\pi_{\mathcal{T}_L}$ and $\nu_{\mathcal{T}_L}^\theta$ the stationary distributions of $\mathbf{CP}_{\rho^+}(\mathcal{T}_L^+)$ and $\mathbf{DP}_{\rho^+}(\mathcal{T}_L^+)$, respectively. Notice that, by Markov's inequality and Lemma 2.4.2, for any $\lambda \leqslant \lambda_0$ we have

$$\mathbb{P}\Big(\nu_{\mathcal{T}_L}^{\theta}(\mathbf{0})^{-1} \geqslant 2m\Big) \leqslant \frac{\mathbb{E}\Big[\nu_{\mathcal{T}_L}^{\theta}(\mathbf{0})^{-1}\Big]}{2m} \leqslant \frac{1}{m}. \tag{2.12}$$

We denote by A_h the set of states in $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$ with depth at least h until the time S_L, i.e.,

$$A_h = \Big\{x \in \{0,1\}^{V(\mathcal{T}_L)} : r\Big(x; \mathcal{T}_L^+\Big) \geqslant h\Big\}.$$

Using the above arguments and the transition rate of the delayed contact process, we deduce the following inequality which holds for all $\lambda \leqslant \lambda_0$ and on the event $\nu_{\mathcal{T}_L}^{\theta}(\mathbf{0})^{-1} \geqslant 2m$,

$$\frac{\pi_{\mathcal{T}_L}(A_h)}{\pi_{\mathcal{T}_L}(\mathbf{0})} = \frac{\sum_{x \in A_h} \theta^{r\left(x;\mathcal{T}_L^+\right)} \nu_{\mathcal{T}_L}^{\theta}(x)}{\nu_{\mathcal{T}_L}^{\theta}(\mathbf{0})} \leqslant \frac{\nu_{\mathcal{T}_L}^{\theta}(A_h)}{\nu_{\mathcal{T}_L}^{\theta}(\mathbf{0})}\theta^h \leqslant 2m\theta^h,$$

By (2.12), this inequality thus holds with probability $1 - m^{-1}$ over the choice of $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$.

On the other hand, if $(X_t) \sim \mathbf{CP}_{\rho^+}(\mathcal{T}_L^+)$ hits A_h, the expected time needed for X_t to escape from A_h is at least 1. Indeed, we must wait for a recovery exponential clock rings. In other words, if

$$\gamma_L(h) := \Big|\{t \in [0, S_L] : X_t \in A_h\}\Big|,$$

where $|\cdot|$ denotes the Lebesgue measure, then

$$\mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[\gamma_L(h) \mid H > h\Big] \geqslant 1.$$

Therefore, for any $\lambda \leqslant \lambda_0$, we obtain

$$\begin{aligned}
\mathbb{P}_{\mathcal{T}_L,\mathbb{F}}\Big(H > h\Big) &\leqslant \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[\gamma_L(h) \mid H > h\Big]\, \mathbb{P}_{\mathcal{T}_L,\mathbb{F}}\Big(H > h\Big) \\
&= \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[\gamma_L(h)\mathbf{1}_{\{H>h\}}\Big] \leqslant \mathbb{E}_{\mathcal{T}_L,\mathbb{F}}\Big[\gamma_L(h)\Big] \\
&\leqslant \frac{\pi_{\mathcal{T}_L}(A_h)}{\pi_{\mathcal{T}_L}(\mathbf{0})} \leqslant 2m(K\lambda)^h,
\end{aligned}$$

with probability $1 - m^{-1}$ over the choice of $(\mathcal{T}_L, \mathbb{F}(\mathcal{T}_L))$. □

We are now ready to move to the proof of the main theorem of this section.

Proof of Theorem 2.2.1. We consider the K and λ_0 as defined in Lemma 2.4.2. Let $\lambda \in (0, \lambda_0]$ so that in particular $K\lambda < 1$. Let δ be any small number and h a constant such that $(K\lambda)^h = \frac{\delta^2}{8}$. Further, denote by E_h the event that the infection set of $\mathbf{CP}(\mathcal{T};\mathbf{1}_\rho)$ does not go deeper that h before dying out. Hence, by Lemma 2.4.3 and setting $m = 4\delta^{-1}$, we obtain that

$$\mathbb{P}(E_h) \geqslant 1 - 2m(K\lambda)^h = 1 - 2m\frac{\delta^2}{8} = 1 - \delta.$$

On the other hand, let R and R_h denote the first time when $\mathbf{CP}(\mathcal{T};\mathbf{1}_\rho)$ and $\mathbf{CP}(\mathcal{T}_h;\mathbf{1}_\rho)$ reaches state $\mathbf{0}$, respectively. Under the event E_h these two times are the same. Thus the above reasoning and Lemma 2.4.1 imply that

$$\mathbb{E}\Big[R \,|\, E_h\Big] = \mathbb{E}\Big[R_h \,|\, E_h\Big] = \frac{\mathbb{E}\Big[R_h \mathbf{1}_{E_h}\Big]}{\mathbb{P}(E_h)} \leqslant \frac{\mathbb{E}\Big[R_h\Big]}{\mathbb{P}(E_h)} \leqslant \frac{e^{1/2}\,\mathbb{E}\Big[\mathcal{F}_\rho^2\Big]^{1/2}}{1-\delta} < \infty.$$

Finally, for $(X_t) \sim \mathbf{CP}(\mathcal{T};\mathbf{1}_\rho)$, note that

$$\begin{aligned} \mathbb{P}\Big(X_t \neq \mathbf{0} \text{ for all } t \geqslant 0\Big) &= \mathbb{P}\Big(R = \infty\Big) \\ &= \mathbb{P}\Big(\{R=\infty\} \cap E_h\Big) + \mathbb{P}\Big(\{R=\infty\} \cap E_h^c\Big). \end{aligned}$$

Furthermore, we have that $\mathbb{P}(\{R = \infty\} \cap E_h) = 0$ and $\mathbb{P}(E_h^c) \leqslant \delta$. Therefore, we get the following inequality which holds for all $\delta > 0$

$$\mathbb{P}\Big(X_t \neq \mathbf{0} \text{ for all } t \geqslant 0\Big) \leqslant \delta.$$

We conclude that for all $\lambda \leqslant \lambda_0$ the process $(X_t) \sim \mathbf{CP}(\mathcal{T};\mathbf{1}_\rho)$ dies out almost surely. □

2.5 Finite Stars

In this section, we show some results for the inhomogeneous contact process on stars which will be used in the proof of Theorem 2.2.2 in the next section. Although any finite graph is eventually trapped in the state of zero infection, the stars are able to maintain the infection for a long time. Here we will show that, if the root of the star is initially infected, the star will keep a large number of infected leaves for a long time with probability very close to 1.

Some of our results in this section are inspired by results obtained by Huang and Durret in [40, Section 2]. However, we had to adapt their arguments to take advantage of the fact that we can have a large fitness value associated to the root of the star.

We start this section by proving a lower bound for the probability to transfer the infection from one vertex to another in a graph consisting of a single path conditionally on the associated fitness values.

Lemma 2.5.1. *Let r be an arbitrary non-negative integer and $f \geqslant 1$ a real number. Let $\mathcal{C}_r$ be a graph consisting of a single path of length r on the vertices $v_0, \ldots, v_r$ with associated fitness values $\mathbb{F}(\mathcal{C}_r) := \{\mathcal{F}_{v_0}, \ldots, \mathcal{F}_{v_r}\}$. Consider $(X_t) \sim \mathbf{CP}(\mathcal{C}_r; \mathbf{1}_{v_0})$ the contact process on $\mathcal{C}_r$ where v_0 is initially infected. Then there exists a constant $\gamma > 0$ such that*

$$\mathbb{P}_{\mathbb{F}}\Big(v_r \in X_{2r}\Big) = \mathbb{P}\Big(v_r \in X_{2r} \mid \mathbb{F}(\mathcal{C}_r), v_0 \in X_0\Big) \geqslant \Big(1 - e^{-\gamma r}\Big) \prod_{i=1}^{r} \frac{\lambda \mathcal{F}_{v_{i-1}} \mathcal{F}_{v_i}}{1 + \lambda \mathcal{F}_{v_{i-1}} \mathcal{F}_{v_i}}.$$

Moreover,

$$\mathbb{P}\Big(v_r \in X_{2r} \mid \mathcal{F}_{v_0} \geqslant f, \mathcal{F}_{v_r} \geqslant f\Big) \geqslant \Big(1 - e^{-\gamma r}\Big) C_{\lambda,f} \left(\frac{\lambda}{\lambda+1}\right)^r,$$

where

$$C_{\lambda,f} := \left(\frac{\lambda+1}{\lambda}\right)^2 \left(\frac{\lambda f}{1+\lambda f}\right)^2. \tag{2.13}$$

Proof. We emphasize here that the notation $\mathbb{P}_{\mathbb{F}}\Big(\,\cdot\,\Big)$ corresponds to the conditional probability on the fitness and also that we start with v_0 initially infected. Let r be an arbitrary non-negative integer. First, we need to establish some appropriate notation. We define the sequence of times $(s_i)_{i \geqslant 0}$ by setting $s_0 = 0$ and for $i \in \{1, \ldots, r\}$ defining

$$s_i := \inf\{s \geqslant s_{i-1} :\ v_{i-1} \text{ recovers or infects } v_i \text{ at time } s\}.$$

Denote $T = \sum_{i=1}^{r} t_i$ where $t_i := s_i - s_{i-1}$. Also denote the events

$$B_i := \{v_{i-1} \text{ infects } v_i \text{ before recovering}\} \qquad \text{and} \qquad B = \bigcap_{i=1}^{r} B_i.$$

We begin by noting that from the definition of the event B and the definition of T we can obtain the following lower bound

$$\mathbb{P}_{\mathbb{F}}\Big(v_r \in X_{2r}\Big) \ \geqslant\ \mathbb{P}_{\mathbb{F}}\Big(B \cap \{T \leqslant 2r\}\Big) = \mathbb{P}_{\mathbb{F}}\Big(\,T \leqslant 2r \mid B\Big)\mathbb{P}_{\mathbb{F}}(B). \tag{2.14}$$

Now, we shall establish lower bounds for the two probabilities on the right-hand side above. Conditioning on the fitness and also on the event $\{v_{i-1} \in X_{s_{i-1}}\}$, we know that the probability that v_{i-1} infects v_i before recovering is given by

$$\mathbb{P}_{\{\mathcal{F}_{v_{i-1}},\mathcal{F}_{v_i}\}}\Big(B_i \mid v_{i-1} \in X_{s_{i-1}}\Big) = \frac{\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}{1+\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}.$$

where here we denote by $\mathbb{P}_{\{\mathcal{F}_{v_{i-1}},\mathcal{F}_{v_i}\}}(\cdot)$ the conditional probability $\mathbb{P}(\cdot \mid \{\mathcal{F}_{v_{i-1}},\mathcal{F}_{v_i}\})$. This implies that

$$\mathbb{P}_{\mathbb{F}}(B) = \prod_{i=1}^{r} \mathbb{P}_{\{\mathcal{F}_{v_{i-1}},\mathcal{F}_{v_i}\}}\Big(B_i \mid v_{i-1} \in X_{s_{i-1}}\Big) = \prod_{i=1}^{r} \frac{\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}{1+\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}. \tag{2.15}$$

On the other hand, by an application of Markov's inequality and by the definition of B and T, for any $\theta > 0$,

$$\begin{aligned}\mathbb{P}_{\mathbb{F}}\Big(T \geqslant 2r \mid B\Big) &= \mathbb{P}_{\mathbb{F}}\Big(e^{\theta T} \geqslant e^{2\theta r} \Bigm| B\Big) \leqslant e^{-2\theta r}\mathbb{E}_{\mathbb{F}}\Big[e^{\theta T} \mid B\Big] \\ &\leqslant e^{-2\theta r} \prod_{i=1}^{r} \mathbb{E}_{\{\mathcal{F}_{v_{i-1}},\mathcal{F}_{v_i}\}}\Big[e^{\theta t_i} \Bigm| B_i \cap \{v_{i-1} \in X_{s_{i-1}}\}\Big]\end{aligned}$$

By conditioning on the event $B_i \cap \{v_{i-1} \in X_{s_{i-1}}\}$ we obtain that t_i has an exponential distribution with parameter $(1+\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i})$. Therefore, we can couple t_i with a random variable τ_i with an exponential distribution with parameter 1 such that $t_i \leqslant \tau_i$ almost surely. Then, following the standard argument for a large deviation bound, we obtain that

$$\mathbb{P}_{\mathbb{F}}\Big(T \geqslant 2r \mid B\Big) \leqslant e^{-2r\theta + r\log\phi(\theta)},$$

where $\phi(\theta) = \mathbb{E}[e^{\theta\tau_i}]$. Now, note that

$$\lim_{\theta\to 0} \frac{\log(\phi_i(\theta))}{\theta} = \mathbb{E}[\tau_i] = 1.$$

Therefore, by choosing $\theta > 0$ small enough, we can deduce that there exists $\gamma > 0$ such that

$$\mathbb{P}_{\mathbb{F}}\Big(T \geqslant 2r \mid B\Big) \leqslant e^{-r\gamma}, \tag{2.16}$$

Plugging (2.15) and (2.16) back into (2.14), we now see that

$$\mathbb{P}_{\mathbb{F}}\Big(v_r \in X_{2r}\Big) \geqslant \Big(1 - e^{-\gamma r}\Big) \prod_{i=1}^{r} \frac{\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}{1+\lambda \mathcal{F}_{v_{i-1}}\mathcal{F}_{v_i}}.$$

For the second part, we fix $f \geqslant 1$. Then, on the event $\{\mathcal{F}_{v_0} \geqslant f, \mathcal{F}_{v_r} \geqslant f\}$ and using that $\mathcal{F}_{v_i} \geqslant 1$, we obtain

$$\prod_{i=1}^{r} \frac{\lambda \mathcal{F}_{v_{i-1}} \mathcal{F}_{v_i}}{1+\lambda \mathcal{F}_{v_{i-1}} \mathcal{F}_{v_i}} \geqslant \left(\frac{\lambda}{1+\lambda}\right)^{r-2} \left(\frac{\lambda f}{1+\lambda f}\right)^{2},$$

which yields the desired result. □

Let $\mathcal{G}_k$ be a star of size k, that is, $\mathcal{G}_k$ consists of a root ρ and k other vertices each connected only to ρ, denoted by $v_1, \ldots, v_k$. Let $(X_t) \sim \mathbf{CP}(\mathcal{G}_k; \mathbf{1}_\rho)$ denote the contact process on $\mathcal{G}_k$ with the root initially infected. Define (X_t^0) to be the modification of (X_t) in such a way that the fitness values for the leaves satisfy $\mathcal{F}_{v_1} = \cdots = \mathcal{F}_{v_k} = 1$ and the fitness of the root is $\mathcal{F}_\rho = f \geqslant 1$. Note that, the contact process (X_t^0) may be considered as a contact process without fitness and with rate parameter $\tilde{\lambda} := \lambda f$. Now, on the event $\{\mathcal{F}_\rho \geqslant f\}$ and taking into account that the random variable $\mathcal{F}$ takes values in $[1, \infty)$, we have

$$\lambda f \leqslant \lambda \mathcal{F}_{v_j} \mathcal{F}_\rho \qquad \text{for every} \quad j = 1, \ldots, k.$$

Thus by monotonicity of the contact process in the rate parameter, as discussed in Section 2.3, we have $X_t^0 \subset X_t$ on the event $\{\mathcal{F}_\rho \geqslant f\}$. Moreover, on a star the dynamics of X_t^0 are the same as that of a standard contact process, but where the infection rate is λf. In particular, we can use some of the results in Section 2 of Huang and Durret [41] (see also [20, Lemma 2.2]) describing the persistence of the infection on a star. Some of the results can be taken over directly, however, others need to be adapted so that we can make full use of the fact that we have the extra flexibility of making f large enough.

When considering the process (X_t^0), write the state of the star as (m, n) where m is the number of infected leaves and $n = 0$ or 1 if the center is healthy or infected, respectively. Throughout, we will write $\Lambda_t^0 \subset X_t^0$ for the set of infected leaves. Also, we will write $\mathbb{P}_{(m,n)}$ if we are conditioning on $X_0^0 = (m, n)$.

As in [20, 41], we will reduce the dynamics to a one-dimensional chain, by concentrating on the first coordinate (i.e. we will count the number of infected leaves). As a first step, we will ignore the times when the centre is not infected and as a second step we will stop the dynamics when we reach a certain level L of infected leaves. We can then define a suitable time-homogenous Markov chain that lower bounds the number of infected leaves (running on a clock ignoring times when the centre is not infected).

To deal with the number of leaves that recover while the root is not infected, we note that when the state is $(m, 0)$ for some $m > 0$, the next event will occur after an exponential time with mean $1/(m\lambda f + m)$. The probability that the root is reinfected first is $\lambda f/(\lambda f + 1)$. Denote by $\mathfrak{N}$ the number of infected leaves that will recover while the center is healthy. Thus $\mathfrak{N}$ has a shifted geometric distribution with success probability $\lambda f/(\lambda f + 1)$, i.e.

$$\mathbb{P}\big(\mathfrak{N} = j\big) = \left(\frac{1}{\lambda f + 1}\right)^j \left(\frac{\lambda f}{\lambda f + 1}\right), \quad j \leqslant k. \tag{2.17}$$

Recall that we are considering a star of size k. Fix $\lambda > 0, k \geqslant 1$ and $f \geqslant 1$ and set a cut-off level

$$L = \left\lceil \frac{\lambda f k}{1 + 2\lambda f} \right\rceil. \tag{2.18}$$

If we modify the chain so that the infection rate is 0 when the number of infected leaves is $\geqslant L$, then we can couple the number of infected leaves to a process $(Y_t)_{t \geqslant 0}$ with the following dynamics

$$\begin{array}{ll} \text{jump} & \text{at rate} \\ Y_t \to Y_t - 1, & L \\ Y_t \to \min\{Y_t + 1, L\}, & \lambda f(k - L) \\ Y_t \to Y_t - \mathfrak{N}, & 1, \end{array} \tag{2.19}$$

so that the process (Y_t) stays below the number of infected leaves (ignoring times when the centre is not infected) as long as the original process has not hit the state $(0, 0)$ yet. For convenience, we do not stop the process after hitting a state below 0 and instead we are careful to apply the coupling only up to the hitting of $(0, 0)$.

The following lemma allows to show that $|\Lambda_t^0|$ hits level L before the process dies out with high probability. Also, we show that the first time (Y_t) hits L has small expectation. Our result is similar to [41, Lemma 2.5] and [20, Lemma 2.3], however we need to adapt their arguments to give useful estimates also for large fitness.

To formalize these statements, denote for the original chain for any $A \geqslant 0$,

$$T_A = \inf\{t \geqslant 0 \,:\, |\Lambda_t^0| \geqslant A\}, \quad T_{0,0} = \inf\{t \geqslant 0 \,:\, X_t^0 = (0, 0)\}.$$

Moreover, for (Y_t) define for $A \geqslant 0$,

$$T_A^Y = \inf\{t \geqslant 0 : Y_t \geqslant A\} \qquad \text{and} \qquad R_0^Y = \inf\{t \geqslant T_1^Y : Y_t \leqslant 0\}.$$

In the following lemma, we will also consider the embedded discrete time process

(Z_n) of (Y_t) obtained by looking at Y_t only at its jump times. This process has the property that for f large enough $(1+\lambda f/2)^{-Z_n}$ is a supermartingale while $Z_n \in (0, L)$. The proof follows from similar arguments as those used in [41, Lemma 2.1].

Lemma 2.5.2. *Let $\lambda > 0$ be fixed. Consider the stochastic process (Y_t) defined in* (2.19) *and the contact process (X_t^0). Then for f and k large enough we have*

$$\mathbb{P}_{(0,1)}\Big(T_L > T_{0,0}\Big) \leqslant \frac{c}{\lambda f k^{1/3}} \qquad \text{and} \qquad \mathbb{E}_0\left[T_L^Y\right] \leqslant \frac{\widehat{c}}{\lambda f},$$

for some positive constants c and $\widehat{c}$.

Remark 2.5.1. Note that [41, Lemma 2.5] also states a result for the conditional expected value of T_L. In their proof the authors ignore times when the root is not infected, so that their result for the expected value is really only for T_L^Y. However, the estimate on the probability is only true for the original process and not for Y_t. We fix this omission by bounding the the times when the root is not infected in the next lemma.

Proof of Lemma 2.5.2. Let $\lambda > 0$, $k \geqslant 1$ and $f \geqslant 1$ and define

$$K = \left\lceil \frac{\lambda (fk)^{1/3}}{1 + 2\lambda f^{1/3}} \right\rceil.$$

Recall the definition of the constant L given in (2.18). Observe that $K \leqslant L$ and $K \to \lceil k^{1/3}/2 \rceil$ as $f \to \infty$. We begin by observing that

$$\mathbb{P}_{(0,1)}\Big(T_K < T_{0,0}\Big) \geqslant \prod_{j=0}^{K-1} \frac{(k-j)\lambda f}{1 + (k-j)\lambda f + j},$$

where the term in the product corresponds to the probability that $|\Lambda_t^0|$ jumps upwards K times before either the root or one of the leaves recovers. From the latter inequality and using [28, Lemma 3.4.3], we have

$$\begin{aligned}
\mathbb{P}_{(0,1)}\Big(T_K > T_{0,0}\Big) &\leqslant \prod_{j=0}^{K-1} 1 - \prod_{j=0}^{K-1} \frac{(k-j)\lambda f}{1 + (k-j)\lambda f + j} \leqslant \sum_{j=0}^{K-1} \frac{1+j}{1 + (k-j)\lambda f + j} \\
&\leqslant \sum_{j=0}^{K-1} \frac{1+j}{(k-j)\lambda f} \leqslant \frac{K^2}{(k-K)\lambda f},
\end{aligned}$$

where in the last inequality we have used that $\{(1+j)/(k-j), j = 0, \ldots, K-1\}$ is an increasing finite sequence. Using the definition of K we can see that

$$\mathbb{P}_{(0,1)}\Big(T_K > T_{0,0}\Big) \leqslant \frac{c_1\lambda(fk)^{2/3}}{(1+2\lambda f^{1/3})(k+2\lambda f^{1/3}k-\lambda(fk)^{1/3})f} \leqslant \frac{c_2}{\lambda f k^{1/3}},$$

where c_1 and c_2 are positive constants.

Now, we use the jump process (Z_n) and the fact that for f large enough the process $(1+\lambda f/2)^{-Z_n}$ is a supermartingale while $Z_n \in (0, L)$. We denote by T_L^Z and R_0^Z the analogous stopping time and return time for the process (Z_n), respectively. Note that, since (Z_n) is obtained from (Y_t) by looking it only at its jump times, we see that $\{T_L^Y > R_0^Y\} = \{T_L^Z > R_0^Z\}$. By an application of the Optional Stopping Theorem with the bounded stopping time $\tau \wedge n$ where $\tau = R_0^Z \wedge T_L^Z$, we get

$$\mathbb{E}_K\Big[\Big(1+\lambda f/2\Big)^{-Z_{\tau\wedge n}}\Big] \leqslant \mathbb{E}_K\Big[\Big(1+\lambda f/2\Big)^{-Z_0}\Big].$$

By letting $n \to \infty$, we deduce

$$\mathbb{E}_K\Big[\Big(1+\lambda f/2\Big)^{-Z_\tau}\Big] \leqslant \Big(1+\lambda f/2\Big)^{-K},$$

which implies

$$\mathbb{P}_K\Big(R_0^Z < T_L^Z\Big) + \Big(1-\mathbb{P}_K\Big(R_0^Z < T_L^Z\Big)\Big)\Big(1+\lambda f/2\Big)^{-L} \leqslant \Big(1+\lambda f/2\Big)^{-K}.$$

Discarding the second term on the left-hand side, it follows that

$$\mathbb{P}_K\Big(R_0^Z < T_L^Z\Big) \leqslant \Big(1+\lambda f/2\Big)^{-K}. \tag{2.20}$$

To go back to the unmodified process, we use that $\{T_{0,0} < T_L\} \subset \{R_0^Z < T_L^Z\}$, so that we can apply (2.20). Combined with the strong Markov property we obtain

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(T_L > T_{0,0}\Big) &\leqslant \mathbb{P}_{(0,1)}\Big(T_K > T_{0,0}\Big) + \mathbb{P}_{(0,1)}\Big(T_L > T_{0,0}\,|\,T_K < T_{0,0}\Big)\mathbb{P}_{(0,1)}\Big(T_K < T_{0,0}\Big)\\ &\leqslant \mathbb{P}_0\Big(T_K^Y > R_0^Y\Big) + \mathbb{P}_{(K,1)}\Big(T_L > T_{0,0}\Big)\\ &\leqslant \mathbb{P}_0\Big(T_K^Y > R_0^Y\Big) + \mathbb{P}_K\Big(R_0^Z < T_L^Z\Big).\end{aligned}$$

Then, combining with the above estimates, we have the first claim of the lemma, that is to say, for f and k large enough

$$\mathbb{P}_{(0,1)}\Big(T_L > T_{0,0}\Big) \leqslant \frac{c_2}{\lambda f k^{1/3}} + \Big(1+\lambda f/2\Big)^{-K} \leqslant \frac{c_3}{\lambda f k^{1/3}},$$

for some positive constant c_3.

To deal with the second claim of the lemma, recall the random variable $\mathfrak{N}$ defined in (2.17) with $\mathbb{E}_0[\mathfrak{N}] = (\lambda f)^{-1}$. From the definition of the process (Y_t), we deduce that before time T_L^Y,

$$\mathbb{E}_0\Big[Y_t\Big] = \Big(- L + \lambda f(k-L) - \frac{1}{\lambda f}\Big)t.$$

Thus the process $(Y_t - \mu_{\lambda,f} t)$ with $\mu_{\lambda,f} := -L + \lambda f(k-L) - (\lambda f)^{-1}$ is a martingale before time T_L^Y. An application of the Optimal Stopping Theorem with the bounded stopping time $T_L^Y \wedge t$, tell us that

$$\mathbb{E}_0\Big[Y_{T_L^Y \wedge t}\Big] - \mu_{\lambda,f}\mathbb{E}_0\Big[T_L^Y \wedge t\Big] = \mathbb{E}_0[Y_0] = 0.$$

Furthermore, since $\mathbb{E}_0[Y_{Y_L^Y \wedge t}] \leqslant L$, it follows that for f and k large enough

$$\mathbb{E}_0\Big[T_L^Y \wedge t\Big] = \frac{L}{\mu_{\lambda,f}} = \frac{1}{-1 + \lambda f k L^{-1} - (\lambda f L)^{-1}} \leqslant \frac{c_4}{\lambda f},$$

Taking $t \to \infty$ we get $\mathbb{E}_0\Big[T_L^Y\Big] \leqslant c_4/\lambda f$. □

By combining the two results of the previous lemma with an estimate on the time that the root of the star is not infected, we can show that the process reaches L infected leaves before time 1 with high probability for large f.

Lemma 2.5.3. *Let $\mathcal{G}_k$ be a star with leaves $v_1, \ldots, v_k$ and root ρ. Consider the contact process $(X_t^0) \sim \mathbf{CP}(\mathcal{G}_k; \mathbf{1}_\rho)$. Let $\lambda > 0$ be fixed. Then for f and k large enough, we have*

$$\mathbb{P}_{(0,1)}\Big(T_L > 1\Big) \leqslant \frac{\widehat{c}_1}{\lambda f} + \frac{c}{\lambda f k^{1/3}}$$

where $\widehat{c}_1$ and c are positive constants.

Proof. Fix $\lambda > 0$. Recall the process (Y_t) given in (2.19) and its corresponding stopping times T_L^Y and R_0^Y. We begin by letting the stochastic process $(\Gamma_t)_{t \geqslant 0}$ denote the number of infected leaves, where we ignore times when the root is not infected. Let us now introduce the stopping time $T_L^\Gamma = \inf\{t > 0 : \Gamma_t \geqslant L\}$. Then, by definition we have $T_L \geqslant T_L^\Gamma$ a.s.. Moreover, on the event that $T_L < T_{0,0}$, we have that $T_L^\Gamma \leqslant T_L^Y$.

Observe that the following inequalities hold

$$\begin{aligned} \mathbb{P}_{(0,1)}\Big(T_L > 1\Big) &\leqslant \mathbb{P}_{(0,1)}\Big(T_L > 1,\ T_L^\Gamma > 1/2\Big) + \mathbb{P}_{(0,1)}\Big(T_L > 1,\ T_L^\Gamma < 1/2\Big) \\ &\leqslant \mathbb{P}_{(0,1)}\Big(T_L^\Gamma > 1/2\Big) + \mathbb{P}_{(0,1)}\Big(T_L > 1,\ T_L^\Gamma < 1/2\Big). \end{aligned} \tag{2.21}$$

Our objective is now to find upper bounds for both probabilities on the right-hand side above. First, we deal with the first of these probabilities. Since we have that

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(T_L^\Gamma > 1/2\Big) &\leqslant \mathbb{P}_{(0,1)}\Big(T_L > T_{0,0}\Big) + \mathbb{P}_{(0,1)}\Big(T_L^\Gamma > 1/2, T_L < T_{0,0}\Big)\\ &\leqslant \frac{c}{\lambda f k^{1/3}} + \mathbb{P}_0(T_L^Y > 1/2\Big),\end{aligned}$$

where we used the first part of Lemma 2.5.2. By the second part of the same lemma and Markov's inequality, we also have that

$$\mathbb{P}_0(T_L^Y > 1/2\Big) \leqslant 2\,\mathbb{E}_0[T_L^Y] \leqslant \frac{2\widehat{c}}{\lambda f}.$$

Combining, we obtain

$$\mathbb{P}_{(0,1)}\Big(T_L^\Gamma > 1/2\Big) \leqslant \frac{c}{\lambda f k^{1/3}} + \frac{2\widehat{c}}{\lambda f}.$$

To estimate the second term in (2.21), let ℓ_t denote the amount of time until t that the process (X_t^0) spends in the states when the root is not infected, i.e.

$$\ell_t = \Big|\{s \leqslant t : X_s^0 = (j,0), \text{ for some } j \in \{0,1,\dots,k\}\Big|,$$

where $|\cdot|$ denotes the Lebesgue measure here. Note that as $T_L = T_L^\Gamma + \ell_{T_L}$, it follows that

$$\mathbb{P}_{(0,1)}\Big(T_L > 1,\ T_L^\Gamma < 1/2\Big) \leqslant \mathbb{P}_{(0,1)}\Big(T_L^\Gamma < 1/2,\ \ell_{T_L} > 1/2\Big). \tag{2.22}$$

Now, we recall the random variable $\mathfrak{N}$ defined in (2.17). Denote by m_t the number of times that the process (Γ_t) jumps down by $\mathfrak{N}$ until time t. Note that if the state is $(i,0)$ with $i \geqslant 1$, then the time until either the next jump down or when the root is infected is given by an exponential random variable with distribution $\mathbf{Exp}(i(1+\lambda f))$. Such a random variable is stochastically dominated by another random variable that has distribution $\mathbf{Exp}(1+\lambda f)$. Hence, each time period when the root is healthy can be dominated by

$$\sum_{i=1}^{\mathfrak{N}} E_i,$$

where for each $i = 1,\dots,\mathfrak{N}$ the random variables E_i have $\mathbf{Exp}(1+\lambda f)$ distributions.

Thus on the event, $\{T_L^\Gamma < 1/2\}$, we deduce that

$$\ell_{T_L} \leqslant \sum_{j=1}^{m_{T_L^\Gamma}} \sum_{i=1}^{\mathfrak{N}_j} E_i^{(j)} \leqslant \sum_{j=1}^{m_{1/2}} \sum_{i=1}^{\mathfrak{N}_j} E_i^{(j)},$$

where the random variables $E_i^{(j)}$ have $\mathbf{Exp}(1+\lambda f)$ distributions and $\mathfrak{N}_j$ are random variables with the same distribution as $\mathfrak{N}$. It follows that

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(T_L^\Gamma < 1/2,\ \ell_{T_L} > 1/2,\ m_{1/2} < \lceil \lambda f\rceil\Big) &\leqslant \mathbb{P}_{(0,1)}\left(\sum_{j=1}^{\lceil \lambda f\rceil} \sum_{i=1}^{\mathfrak{N}_j} E_i^{(j)} > \frac{1}{2}\right)\\ &\leqslant 2\lceil \lambda f\rceil \mathbb{E}_{(0,1)}\Big[\mathfrak{N}\Big]\mathbb{E}_{(0,1)}\Big[E_1\Big]\\ &\leqslant 2(\lambda f+1)\frac{1}{\lambda f}\frac{1}{1+\lambda f} = 2\frac{1}{\lambda f}.\end{aligned}$$

In addition, note that the random variable $m_{1/2}$ has a Poisson distribution with parameter $1/2$. Then appealing to Markov's inequality, we obtain

$$\mathbb{P}_{(0,1)}\Big(m_{1/2} > \lceil \lambda f\rceil\Big) \leqslant \frac{1}{\lceil \lambda f\rceil}\mathbb{E}_{(0,1)}[m_{1/2}] \leqslant \frac{1}{2\lambda f}.$$

Therefore

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(T_L > 1,\ \ell_{T_L} < 1/2\Big) &\leqslant \mathbb{P}(m_{1/2} > \lceil \lambda f\rceil) + \mathbb{P}_{(0,1)}\Big(T_L > 1,\ \ell_{T_L} < 1/2, m_{1/2} < \lceil \lambda f\rceil\Big)\\ &\leqslant \frac{5}{2\lambda f}.\end{aligned}$$

Putting the pieces together back into (2.21), we obtain

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(T_L > 1\Big) &\leqslant \mathbb{P}_{(0,1)}\Big(T_L^\Gamma > 1/2\Big) + \mathbb{P}_{(0,1)}\Big(T_L > 1,\ T_L^\Gamma < 1/2\Big)\\ &\leqslant \frac{2\widehat{c}}{\lambda f} + \frac{c}{\lambda f k^{1/3}} + \frac{5}{2\lambda f} \leqslant \frac{\widehat{c}_1}{\lambda f} + \frac{c}{\lambda f k^{1/3}},\end{aligned}$$

for some positive constant $\widehat{c_1}$. This concludes the proof. □

The following lemma, proved by Huang and Durrett in [41], will be useful for our next result.

Lemma 2.5.4 ([41, Lemma 2.4]). *Let k be an arbitrary non-negative integer and $f \geqslant 1$ a real number. Let $\mathcal{G}_k$ be a star with leaves $v_1, \dots, v_k$ and root ρ. Consider the contact process $(X_t^0) \sim \mathbf{CP}(\mathcal{G}_k; \mathbf{1}_{\{\rho, v_1, \dots, v_L\}})$ where ρ and $L = \lceil \lambda f k/(1+2\lambda f)\rceil$ leaves* *are initially infected. Then, for any $\epsilon \in (0,1/2)$, we have*

$$\mathbb{P}_{(L,1)}\left(\inf_{0\leqslant t\leqslant S}|X_t^0|\leqslant \epsilon L\right)\leqslant (3+\lambda f)(1+\lambda f/2)^{-\epsilon L},$$

where

$$S=\frac{(1+\lambda f/2)^{L(1-2\epsilon)}}{2k(2+\lambda f)}=\frac{1}{4k}(1+\lambda f/2)^{L(1-2\epsilon)-1}. \tag{2.23}$$

The next lemma tell us that if f and k are large enough, then beginning with only vertex ρ infected at time 0, the number of infected leaves during the time interval $[1,S]$ is at leasts ϵL with high probability. We recall that $\Lambda_t \subset X_t$ is the set of infected neighbours of ρ at time t.

Lemma 2.5.5. *Let $\mathcal{G}_k$ be a star of size k with root ρ. Consider $(X_t)\sim \mathbf{CP}(\mathcal{G}_k;\mathbf{1}_\rho)$ the inhomogeneous contact process on $\mathcal{G}_k$. Fix $\lambda>0$, then for any $\epsilon\in(0,1/2)$, we have for f and k large enough, that*

$$\mathbb{P}_{(0,1)}\left(\inf_{1\leqslant t\leqslant S}|\Lambda_t|\leqslant \epsilon L \;\middle|\; \mathcal{F}_\rho\geqslant f\right)\leqslant R(f,k,\lambda),$$

where

$$R(f,k,\lambda)=\frac{\widehat{c}_1}{\lambda f}+\frac{c}{\lambda f k^{1/3}}+\frac{c_2}{fk} \tag{2.24}$$

with $c,\widehat{c}_1$ and c_2 are positive constants.

Proof. Fix $\lambda>0$. We begin by noting that on the event $\{\mathcal{F}_\rho\geqslant f\}$ and by monotonicity we have

$$\left\{\inf_{1\leqslant t\leqslant S}|\Lambda_t|\leqslant \epsilon L\right\}\subset\left\{\inf_{1\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L\right\}.$$

Then it is enough to prove the estimate for the process (Λ_t^0). By the strong Markov property applied at T_L on the event that $T_L<1$ and using that at T_L the root is necessarily infected, we note that

$$\begin{aligned}\mathbb{P}_{(0,1)}\left(\inf_{1\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L\right)&=\mathbb{P}_{(0,1)}\left(\inf_{1\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L,\ T_L\geqslant 1\right)\\&\quad+\ \mathbb{P}_{(0,1)}\left(\inf_{1\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L,\ T_L< 1\right)\\&\leqslant \mathbb{P}_{(0,1)}\Big(T_L\geqslant 1\Big)+\mathbb{P}_{(L,1)}\left(\inf_{0\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L\right).\end{aligned} \tag{2.25}$$

Now, appealing to Lemma 2.5.4, we have for any $\epsilon\in(0,1/2)$, and for f and k large enough,

$$\mathbb{P}_{(L,1)}\left(\inf_{0\leqslant t\leqslant S}|\Lambda_t^0|\leqslant \epsilon L\right)\leqslant \frac{c_2}{fk},$$

for some $c_2 > 0$. Plugging this back into (2.25) and using Lemma 2.5.3, we get the desired result. $\square$

One more lemma will be needed for the next section, so we record it now. The result tells us about the behaviour of the contact process on a graph consisting of a star with a single path joined to one of its leaves. Roughly speaking, if we start with the root of the star infected, we give a lower bound for the probability that the vertex on the path that is furthest from the root will be infected. This is a similar result to [41, Lemma 3.2].

Lemma 2.5.6. *Let $r, k \geqslant 1$ be arbitrary integers, and $f \geqslant 1$ a real number. Let $\mathcal{G}_k$ be the star of size k with root ρ and leaves $v_1, \dots, v_k$, to which has been added a single path of length r of descendants of some child v_i of ρ. Denote by $\mathcal{C}_r$ the path with vertices $u_1, \dots, u_r$ with $u_1 = v_i$ and associated fitness values $\mathbb{F}(\mathcal{C}_r) = \{\mathcal{F}_{u_1}, \dots, \mathcal{F}_{u_r}\}$. Consider $(X_t) \sim \mathbf{CP}(\mathcal{G}_k \cup \mathcal{C}_r; \mathbf{1}_\rho)$ the inhomogeneous contact process on $\mathcal{G}_k \cup \mathcal{C}_r$ where ρ is initially infected. Then, for f, k and r large enough, we have*

$$\mathbb{P}_{(0,1)}\Big(u_r \notin X_s \text{ for all } s \in [0, S] \;\Big|\; \mathcal{F}_{u_r} \geqslant f, \mathcal{F}_\rho \geqslant f\Big) \leqslant \Big(1 - \widehat{C}_{\lambda,f}\widehat{\lambda}^r\Big)^{S/(2r+1)} + R(f, k, \lambda),$$

where

$$\widehat{C}_{\lambda,f} = \frac{1}{4} C_{\lambda,f} \qquad \text{and} \qquad \widehat{\lambda} = \frac{\lambda}{\lambda + 1}. \tag{2.26}$$

The terms S, $C_{\lambda,f}$ and $R(f,k,\lambda)$ were defined in (2.23), (2.13) *and* (2.24), *respectively.*

Proof. Let $r \geqslant 1$ be an arbitrary integer and $m \leqslant S(2r+1)^{-1}$. Begin by noting that

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(u_r \notin X_s \text{ for all } s \in [0, S] \;\Big|\; \mathcal{F}_\rho \geqslant f, \mathcal{F}_{u_r} \geqslant f\Big) &\leqslant \mathbb{P}_{(0,1)}\Big(\inf_{1 \leqslant t \leqslant S} |\Lambda_t| \leqslant \epsilon L \;\Big|\; \mathcal{F}_\rho \geqslant f\Big) \\ + \mathbb{P}_{(0,1)}\Big(u_r \notin X_s \text{ for all } s \in [0, m(2r+1)] \;\Big|\; &\inf_{1 \leqslant s \leqslant S} |\Lambda_s| \geqslant \epsilon L,\ \mathcal{F}_\rho \geqslant f, \mathcal{F}_{u_r} \geqslant f\Big),\end{aligned}$$

where we recall that $|\Lambda_t|$ is the number of infected leaves of ρ at time t. From Lemma 2.5.5 we know for f and k large enough that

$$\mathbb{P}_{(0,1)}\Big(\inf_{1 \leqslant t \leqslant S} |\Lambda_t| \leqslant \epsilon L \;\Big|\; \mathcal{F}_\rho \geqslant f\Big) \leqslant R(f, k, \lambda),$$

where $R(f,k,\lambda)$ was defined in (2.24). The proof is thus complete as soon as we can show that for f and r sufficiently large

$$\begin{aligned}\mathbb{P}_{(0,1)}\Big(u_r \notin X_s \text{ for all } s \in [0, m(2r+1)] \;\Big|\; \inf_{1 \leqslant s \leqslant S} |\Lambda_s| \geqslant \epsilon L,\ \mathcal{F}_\rho \geqslant f, \mathcal{F}_{u_r} \geqslant f\Big) \\ \leqslant \Big(1 - \widehat{C}_{\lambda,f}\widehat{\lambda}^r\Big)^m.\end{aligned} \tag{2.27}$$

We work on the event $\{\inf_{1\leqslant s\leqslant S}|\Lambda_s| \geqslant \epsilon L,\ \mathcal{F}_\rho \geqslant f, \mathcal{F}_{u_r} \geqslant f\}$. Define the sequence of times $\{t_i = (2r+1)i,\ 1 \leqslant i \leqslant m\}$. Fix $i \in \{1,\dots,m\}$. We shall find the probability that u_r is infected in the time interval $[t_i, t_{i+1})$. Note that the center ρ is not necessarily infected at time t_i. However, since $1 \leqslant t_i \leqslant S$, it follows that the number of infected neighbours at time t_i is at least ϵL. Thus the center ρ will be infected by time $t_i + 1$ with probability

$$\geqslant 1 - \exp\Big(- \lambda f \epsilon L\Big).$$

Now, by Lemma 2.5.1 we have that if ρ is infected at time $t_i + 1$ then, under the event $\{\mathcal{F}_\rho \geqslant f, \mathcal{F}_{u_r} \geqslant f\}$, the vertex u_r will become infected at time t_{i+1} (observe that $2r = t_{i+1} - (t_i + 1)$) with probability $\geqslant (1 - e^{-r\gamma})C_{\lambda,f}\left(\frac{\lambda}{1+\lambda}\right)^r$ for some $\gamma > 0$. Hence, the probability that u_r is successfully infected in $[t_i, t_{i+1})$ is

$$\geqslant (1 - e^{-\lambda f \epsilon L})(1 - e^{-r\gamma})C_{\lambda,f}\left(\frac{\lambda}{1+\lambda}\right)^r.$$

Taking r and f sufficiently large, then the latter expression is $\geqslant \frac{1}{4}C_{\lambda,f}\widehat{\lambda}^r$, so it follows that (2.27) holds. Thus the desired result follows. $\square$

2.6 Proof of Theorem 2.2.2

For the proof of Theorem 2.2.2, we need the following two lemmas, which the reader can find in [64, Lemma 2.3 and Lemma 3.4]. The first lemma gives a lower bound for the probability that a binomial random variable is at least 1. The second lemma gives a necessary condition for $\liminf_{t\to\infty} g(t)$ to be positive, where g is a function on the non-negative reals numbers. From the definition of λ_2, we can see that it will be convenient to apply this result to the function $g(t) = \mathbb{P}(\rho \in X_t)$.

Lemma 2.6.1 ([64, Lemma 2.3]). *Let M be a positive integer-valued random variable and pick $p < \mathbb{E}[M]$. For any $x \in (0,1]$, let M_x be a random variable with binomial distribution* $\mathbf{Bin}(M,x)$. Then there exits $\epsilon > 0$ such that

$$\mathbb{P}(M_x \geqslant 1) \geqslant px \wedge \epsilon.$$

Lemma 2.6.2 ([64, Lemma 3.4]). *Let G be any non-decreasing function on $[0,\infty)$ such that $G(x) \geqslant x$ on some neighbourhood of 0. Suppose g is a function on $[0,\infty)$*

that satisfies, for some $S > 0$,

$$\inf_{0 \leqslant t \leqslant S} g(t) > 0 \qquad \text{and} \qquad g(t) \geqslant G\left(\inf_{0 \leqslant s \leqslant t-S} g(s)\right) \qquad \text{for} \quad t > S. \tag{2.28}$$

Then

$$\liminf_{t\to\infty} g(t) > 0.$$

Remark 2.6.1. To prove Theorem 2.2.2 it suffices to show that for any $\lambda > 0$ we have $\liminf_{t\to\infty} \mathbb{P}(\rho \in X_t) > 0$ without conditioning on the tree $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$. To see why, assume that $\liminf_{t\to\infty} \mathbb{P}(\rho \in X_t) > 0$ for any $\lambda > 0$. Now, an application of the dominated convergence theorem yields

$$\mathbb{E}\left[\liminf_{t\to\infty} \mathbb{P}_{\mathcal{T},\mathbb{F}}(\rho \in X_t)\right] = \liminf_{t\to\infty} \mathbb{E}\Big[\mathbb{P}_{\mathcal{T},\mathbb{F}}(\rho \in X_t)\Big] > 0.$$

It follows that for all $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ from a set with positive probability we deduce that

$$\liminf_{t\to\infty} \mathbb{P}_{\mathcal{T},\mathbb{F}}(\rho \in X_t) > 0,$$

and this in turn implies $\lambda_2(\mathcal{T}, \mathbb{F}(\mathcal{T})) = 0$. By using the same arguments as in Proposition 3.1 in [64] we see that $\lambda_2(\mathcal{T}, \mathbb{F}(\mathcal{T})) = 0$ is constant for a.e. $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ conditioned on $|\mathcal{T}| = \infty$. It is worth recalling here that we are working under the assumption $\mu = \mathbb{E}[\xi] > 1$, which guarantees that $\mathcal{T}$ survives forever with positive probability.

The proof of our result follows from similar ideas as those used in [64, Theorem 3.2] and afterwards in [41, Theorem 1.4]. However, the presence of fitness turns out to lead to significant changes throughout the whole proof.

Proof of Theorem 2.2.2. Throughout this proof fix $\lambda > 0$. We aim to show that under the hypotheses of the theorem, the contact process always survives strongly.

In the proof, we will use two parameters: a positive integer k and a real number $f \geqslant 1$ that we will choose at the very end. Let the set of *good vertices* in $(\mathcal{T}, \mathbb{F}(\mathcal{T}))$ be denoted by

$$A_{f,k} := \Big\{ v \in V(\mathcal{T}) : \ \mathcal{F}_v \geqslant f \quad \text{and} \quad \#\text{chil}(v) = k \Big\},$$

where $\#\text{chil}(v)$ is the number of children of vertex v. Assume that k and $f \geqslant 1$ are chosen so that the event $\{\rho \in A_{f,k}\}$ has strictly positive probability. Since we are only interested in proving a lower bound with positive probability, we will from now on assume that $\mathbb{P}$ refers to the conditional probability measure given $\{\rho \in A_{f,k}\}$. Also, we emphasize that we always start with the root initially infected (unless specified otherwise).

The proof is long and we break it up into three steps. In the first step, we will push the infection to good vertices in a suitably chosen generation. In the second and third step, we will bring the infection back to the root appealing to Lemma 2.6.2.

Step 1. Let S, $\widehat{C}_{\lambda,f}$, and $\widehat{\lambda}$ be given as (2.23) and (2.26), respectively. Let r be a positive integer. We assume that $r= r(f,k)$ is a parameter to be determined later which satisfies

$$\frac{S}{2r+1} > \frac{2}{\widehat{C}_{\lambda,f}\widehat{\lambda}^r}. \tag{2.29}$$

Denote by $V_r = \{v \in V(\mathcal{T}) : d(\rho, v) = r\}$, where we recall that $d(\cdot,\cdot)$ is the graph distance between two vertices in the tree. We prove in this step that, conditionally on $v \in V_r$ being a good vertex, then it will be infected before time S with probability bounded away from zero uniformly for f and k large enough. That is to say, there exists a positive constant c_1 such that for any f and k sufficiently large and any $v \in A_{f,k} \cap V_r$,

$$p_{in} := \mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(v \in X_s \text{ for some } s \in [0, S]\ \Big) \geqslant c_1.$$

The proof of this bound follows by an application of Lemma 2.5.6. Indeed, for f sufficiently large we obtain

$$1 - p_{in} \leqslant \Big(1 - \widehat{C}_{\lambda,f}\widehat{\lambda}^r\Big)^{S/(2r+1)} + R(f,k,\lambda).$$

where the function $R(f,k,\lambda)$ is defined in (2.24). Then, using the inequality $(1 - x)^{1/x} < e^{-1}$, observe that (2.29) forces the first term on the right hand side of the last equation to be at most e^{-2}, i.e.

$$\Big(1 - \widehat{C}_{\lambda,f}\widehat{\lambda}^r\Big)^{S/(2r+1)} \leqslant \Big(1 - \widehat{C}_{\lambda,f}\widehat{\lambda}^r\Big)^{2/(\widehat{C}_{\lambda,f}\widehat{\lambda}^r)} < e^{-2}.$$

Further, for f and k sufficiently large we have $R(f,k,\lambda) \leqslant 1/2$. Therefore, we deduce that p_{in} is bounded away from 0 for f and k large enough, i.e.,

$$p_{in} \geqslant c_1, \qquad \text{where} \qquad c_1 = \frac{1}{2} - e^{-2} > 0. \tag{2.30}$$

In other words, with positive probability we push the infection to a generation r which satisfies condition (2.29).

Step 2. We assume that condition (2.29) holds. Let us define, for $t > 0$,

$$g(t) := \mathbb{P}\Big(\rho \in X_t\Big).$$

We want to bring the infection back to the root at time t appealing to Lemma 2.6.2. We need to find a non-decreasing function G such that $G(x) \geqslant x$ on some neighbourhood of 0 and to verify that the function g satisfies (2.28), for S as chosen in (2.23). First, observe that

$$\inf_{0 \leqslant s \leqslant S} g(s) \geqslant \mathbb{P}\Big(\rho \text{ never recovers in } [0, S]\Big) = e^{-S} > 0.$$

Thus, we have that g satisfies the first condition of Lemma 2.6.2. On the other hand, conditioning on the event that $v \in X_{t-s}$ for some $v \in A_{f,k} \cap V_r$, we deduce the following lower bound, for $t > S$

$$g(t) = \mathbb{P}\Big(\rho \in X_t\Big) \geqslant H_1(t) H_2(t), \tag{2.31}$$

where the functions H_1 and H_2 are given by

$$H_1(t) = \mathbb{P}\Big(v \in X_{t-S} \text{ for some } v \in A_{f,k} \cap V_r\Big),$$

$$H_2(t) = \mathbb{P}\Big(\rho \in X_t \;\Big|\; v \in X_{t-S} \text{ for some } v \in A_{f,k} \cap V_r\Big).$$

Hence, the next goal is to establish lower bounds for the functions H_1 and H_2.

Lower bound for H_1. Denote by $J_r = |A_{f,k} \cap V_r|$ the number of good vertices in generation r. Taking into account that we are working conditionally on the event $\{\rho \in A_{f,k}\}$, we deduce a lower bound of the expected number of good vertices at level r,

$$\mathbb{E}[J_r] \geqslant k\mu^{r-1} \mathbb{P}\left(\xi = k\right) \mathbb{P}\left(\mathcal{F} \geqslant f\right),$$

where we recall that $\mu = \mathbb{E}[\xi] > 1$. Let M_r^S be the random number of good vertices in generation r that are infected before time S. Together with (2.30), the above estimate is sufficient to obtain that,

$$\begin{aligned} \mathbb{E}\Big[M_r^S\Big] &\geqslant \mathbb{E}\left[\sum_{v \in A_{f,k} \cap V_r} \mathbb{P}_{\mathcal{T},\mathbb{F}}\Big(v \in X_s \text{ for some } s \in [0, S]\Big)\right] && (2.32) \\ &\geqslant c_1 \mathbb{E}[J_r] \geqslant c_1 k\mu^{r-1} \mathbb{P}\left(\xi = k\right) \mathbb{P}\left(\mathcal{F} \geqslant f\right). && (2.33) \end{aligned}$$

Let us define, for $t > 2S$,

$$\chi(t) := \inf \Big\{ g(s) : 0 \leqslant s \leqslant t - S \Big\}.$$

Now, ignore all the infections of $v \in A_{f,k} \cap V_r$ by its parent except the first infection. Then, the contact on the subtrees rooted at vertices $v \in A_{f,k} \cap V_r$ which are infected at some time $s < S$ will evolve independently from time s to time $t - S$ and then vertex v will be infected with probability at least $\chi(t)$. Therefore, if we denote by M_r^{t-S} the random number of good vertices in generation r that are infected at time $t - S$, we can conclude that the random variable M_r^{t-S} stochastically dominates a random variable M_χ that has distribution $\mathbf{Bin}\Big(M_r^S, \chi(t)\Big)$. By Lemma 2.6.1 and (2.33), there exists $\epsilon_1 > 0$ such that

$$\mathbb{P}\Big(M_\chi \geqslant 1\Big) \geqslant 2^{-1} c_1 k \mu^{r-1} \mathbb{P}\left(\xi = k\right) \mathbb{P}\left(\mathcal{F} \geqslant f\right) \chi(t) \wedge \epsilon_1.$$

Note that the factor 2^{-1} is required to guarantee the hypotheses of the Lemma 2.6.1. Moreover, since M_r^{t-S} dominates the random variable M_χ, we obtain that, for $t > 2S$

$$H_1(t) \geqslant \mathbb{P}\Big(M_r^{t-S} \geqslant 1\Big) \geqslant \mathbb{P}\Big(M_\chi \geqslant 1\Big),$$

which implies for $t > 2S$

$$H_1(t) \geqslant 2^{-1} c_1 k \mu^{r-1} \mathbb{P}\left(\xi = k\right) \mathbb{P}\left(\mathcal{F} \geqslant f\right) \chi(t) \wedge \epsilon_1. \tag{2.34}$$

Lower bound for H_2. Let $t > S$. We break down $H_2(t)$ into a product of conditional probabilities as follows

$$H_2(t) = \mathbb{P}\Big(\rho \in X_t \;\Big|\; v \in X_{t-S} \text{ for some } v \in A_{f,k} \cap V_r\Big) \geqslant h(t)\widehat{h}(t),$$

where

$$\begin{aligned} h(t) &= \mathbb{P}\Big(\rho \in X_s \text{ for some } s \in [t-S, t] \;\Big|\; v \in X_{t-S} \text{ for some } v \in A_{f,k} \cap V_r\Big), \\ \widehat{h}(t) &= \mathbb{P}\Big(\rho \in X_t \;\Big|\; \rho \in X_s \text{ for some } s \in [t-S, t],\; v \in X_{t-S} \text{ for some } v \in A_{f,k} \cap V_r\Big). \end{aligned}$$

We can lower bound $h(t)$ by ignoring all the possible infections other than the infection of v at time $t - S$, we have

$$h(t) \geqslant \mathbb{P}_{\{v\}}\Big(\rho \in X_s \text{ for some } s \leqslant S \;\Big|\; v \in A_{f,k} \cap V_r\Big),$$

where $\mathbb{P}_{\{v\}}$ denote the law of the process with v initially infected. Moreover, by ignoring one child of v and considering the star of size k centered at v, the same argument as in Step 1 applies and we deduce that

$$h(t) \geqslant \mathbb{P}_{\{v\}}\Big(\rho \in X_s \text{ for some } s \leqslant S \;\Big|\; v \in A_{f,k} \cap V_r\Big) \geqslant c_1,$$

where the constant c_1 is the same as in Step 1 and comes from assumption (2.29).

On the other hand, once again by monotonicity of the contact process we have, for $t > S$

$$\widehat{h}(t) \geqslant \mathbb{P}\Big(\rho \in X_t \;\Big|\; \rho \in X_s \text{ for some } s \in [t-S, t]\Big).$$

Let us now introduce the stopping time for X,

$$\tau = \inf\{u \geqslant t - S : \; \rho \in X_u\}.$$

On the event $\{\tau \leqslant t\}$ we may distingusih two events $\{\tau \leqslant t-2\}$ and $\{\tau \in [t-2, t]\}$. We first work on the event $\{\tau \leqslant t-2\}$ and at the end we come back to the other one. Fix $\epsilon_2 > 0$. Let $\mathcal{B}_\tau$ be the event that the number of infected neighbours of ρ is at least $\epsilon_2 L$ in the entire random time interval $[\tau+1, t-1]$, i.e.,

$$\mathcal{B}_\tau = \left\{ \inf_{u \in [\tau+1, t-1]} |\Lambda_u| \geqslant \epsilon_2 L \right\},$$

where we recall that Λ_u is the set of infected leaves of ρ at time u. Denote by $\mathscr{F}_t$ the σ-algebra generated by tree, fitness and the contact process up to time t. Then, appealing to the strong Markov property and Lemma 2.5.5 and using that $\tau \geqslant t - S$, we obtain

$$\mathbb{P}(\mathcal{B}_\tau | \mathscr{F}_\tau) \geqslant \mathbb{P}\left(\inf_{u \in [\tau+1, S+\tau]} |\Lambda_t| \geqslant \epsilon_2 L \;\middle|\; \mathscr{F}_\tau \right) \geqslant 1 - R(f, k, \lambda),$$

where the function $R(f, k, \lambda)$ is defined in (2.24). Next, define $\mathcal{I}$ to be the event that at least one of the $\epsilon_2 L$ neighbours that is infected at time $t-1$ infects the root at a time in $[t-1, t]$ before recovering. We can estimate the probability of this event as

$$\begin{aligned} \mathbb{P}(\mathcal{I} \mid \mathscr{F}_{t-1}) &= 1 - \mathbb{P}(\text{all } \epsilon_2 L \text{ neighbors recover before infecting the root in } [t-1, t]) \\ &\geqslant 1 - \left(\frac{1}{1 + \lambda f} \right)^{\epsilon_2 L}. \end{aligned}$$

Also, note that $\mathbb{P}(\mathcal{R}_\rho | \; \mathscr{F}_{t-1}) \geqslant e^{-1}$, where $\mathcal{R}_\rho = \{\rho \text{ does not recover in } [t-1, t]\}$.

With this notation we have the following estimate

$$\begin{aligned}\widehat{h}(t) \geqslant \mathbb{P}\Big(\mathcal{I} \cap \mathcal{R}_\rho \cap \mathcal{B}_\tau \cap \{\tau \leqslant t-2\}\Big) \\ + \mathbb{P}\Big(\{\rho \in X_s \text{ for all } s \in [\tau, t]\} \cap \{\tau \in [t-2, t]\}\Big).\end{aligned}$$

Furthermore, under the event $\{\tau \in [t-2, t]\}$ we have $t - \tau \leqslant 2$ and thus

$$\mathbb{P}\Big(\{\rho \in X_s \text{ for all } s \in [\tau, t]\} \cap \{\tau \in [t-2, t]\}\Big) \geqslant e^{-2}\mathbb{P}\Big(\tau \in [t-2, t]\Big).$$

Conditioning on $\mathscr{F}_{t-1}$ and using the independence of the infection and recovery events, we obtain that

$$\begin{aligned}\mathbb{P}\Big(\mathcal{I} \cap \mathcal{R}_\rho \cap \mathcal{B}_\tau \cap \{\tau \leqslant t-2\}\Big) &= \mathbb{E}\Big[\mathbb{P}(\mathcal{I} \cap \mathcal{R}_\rho | \mathscr{F}_{t-1}) \mathbf{1}_{\{\mathcal{B}_\tau, \tau \leqslant t-2\}}\Big] \\ &= \mathbb{E}\Big[\mathbb{P}(\mathcal{I}|\mathscr{F}_{t-1})\mathbb{P}(\mathcal{R}_\rho|\mathscr{F}_{t-1})\mathbf{1}_{\{\mathcal{B}_\tau, \tau \leqslant t-2\}}\Big] \\ &\geqslant \left(1 - \left(\frac{1}{1+\lambda f}\right)^{\epsilon_2 L}\right) e^{-1}\mathbb{E}\Big[\mathbb{P}(\mathcal{B}_\tau|\mathscr{F}_\tau)\mathbf{1}_{\{\tau \leqslant t-2\}}\Big].\end{aligned}$$

Combining with the previous estimates, we get the lower bound

$$\begin{aligned}\widehat{h}(t) &\geqslant e^{-2}\mathbb{P}\Big(\tau \in [t-2, t]\Big) + e^{-1}\Big(1 - R(f, \lambda, k)\Big)\left(1 - \left(\frac{1}{1+\lambda f}\right)^{\epsilon_2 L}\right)\mathbb{P}(\tau \leqslant t-2) \\ &\geqslant e^{-2}\Big(1 - R(f, \lambda, k)\Big)\left(1 - \left(\frac{1}{1+\lambda f}\right)^{\epsilon_2 L}\right).\end{aligned}$$

Denote by c_2 the expression

$$c_2 = c_2(\lambda, f, k) := e^{-2}\left(1 - \left(\frac{1}{1+\lambda f}\right)^{\epsilon_2 L}\right)\Big(1 - R(f, k, \lambda)\Big).$$

Observe that this expression depends on λ, f and k, however at the end of the proof we will take $f \to \infty$ and $k \to \infty$ and we will see that c_2 converges to e^{-2} when taking those limits. The above lower bounds for h and $\widehat{h}$ give, for $t > S$

$$H_2(t) \geqslant h(t)\widehat{h}(t) \geqslant c_1 c_2. \tag{2.35}$$

Plugging (2.34) and (2.35) back into (2.31), we now see that, for $t > 2S$

$$g(t) \geqslant H_1(t)H_2(t) \geqslant ck\mu^{r-1}\mathbb{P}\left(\xi = k\right)\mathbb{P}\left(\mathcal{F} \geqslant f\right)\chi(t) \wedge \epsilon_1,$$

where $c = 2^{-1}c_1^2 c_2$. Hence, under the assumption that r satisfies condition (2.29), we establish the following lower bound

$$g(t) \geqslant \begin{cases} ck\mu^{r-1}\mathbb{P}\left(\xi = k\right)\mathbb{P}\left(\mathcal{F} \geqslant f\right)\chi(t) \wedge \epsilon_1, & t > 2S \\ \\ \inf\limits_{0\leqslant s\leqslant 2S} g(s), & S \leqslant t \leqslant 2S. \end{cases}$$

Furthermore, note that

$$\inf_{0\leqslant s\leqslant 2S} g(s) \geqslant \mathbb{P}\Big(\rho \text{ never recovers in } [0, 2S]\Big) = e^{-2S} > 0.$$

Therefore, the above two estimate are sufficient to deduce that there exists $\epsilon > 0$ such that

$$g(t) \geqslant ck\mu^{r-1}\mathbb{P}\left(\xi = k\right)\mathbb{P}\left(\mathcal{F} \geqslant f\right)\chi(t) \wedge \epsilon, \qquad \text{for} \quad t > S.$$

Step 3. We would like to use Lemma 2.6.2 applied to the non-negative and non-decreasing function G defined as

$$G(x) = ck\mu^{r-1}\mathbb{P}\left(\xi = k\right)\mathbb{P}\left(\mathcal{F} \geqslant f\right)x \wedge \epsilon, \qquad \text{for} \quad x \geqslant 0.$$

We begin by noting that if we choose

$$r = r(f,k) = \left\lceil -\frac{\log\Big(\mu^{-1}ck\mathbb{P}(\xi = k)\mathbb{P}(\mathcal{F} \geqslant f)\Big)}{\log\mu}\right\rceil. \tag{2.36}$$

The reason for this choice is that then we will have $ck\mu^{r-1}\mathbb{P}\left(\xi = k\right)\mathbb{P}\left(\mathcal{F} \geqslant f\right) \geqslant 1$, and this in turn implies that $G(x) \geqslant x$ for some neighbourhood of 0. This will allow us to apply Lemma 2.6.2. We have to now show that the above choice of r allows us to choose f and k such that condition (2.29) holds for λ as fixed at the beginning. Since $C_{\lambda,f} \geqslant 1$, we see from the definition of S given in (2.23), that

$$\frac{1}{4}\left(1 + \frac{\lambda f}{2}\right)^{L(1-2\epsilon)-1}\left(\frac{\lambda}{\lambda+1}\right)^{r} > 8k(2r+1),$$

implies condition (2.29). The latter inequality is equivalent to

$$\log\left(\frac{\lambda}{1+\lambda}\right) > \frac{1}{r}\left(\log\left(\frac{32}{3}k(2r+1)\right) - \left(\frac{L}{2} - 1\right)\log\left(1 + \frac{\lambda f}{2}\right)\right). \tag{2.37}$$

In the last part of the proof, we complete our argument by showing that under the hypotheses (**B**) and (**C**) in Theorem 2.2.2 for the fitness and the offspring distributions, we can find f and k such that (2.37) holds. Therefore, we can then deduce that the assumptions of Lemma 2.6.2 are satisfied and so

$$\liminf_{t\to\infty} \mathbb{P}(\rho \in X_t) > 0,$$

which means that the process survives strongly.

(i) We first assume that condition (**C**) in Theorem 2.2.2 holds. Note that from the definition of $r = r(f,k)$ we obtain that for fixed k,

$$\lim_{f\to\infty} \frac{1}{r} \log\left(\frac{32}{3}k(2r+1)\right) = \lim_{r\to\infty} \frac{1}{r} \log\left(\frac{32}{3}k(2r+1)\right) = 0.$$

Now returning to (2.37), taking into account that by the definition of L given in (2.18), $L \to \lceil \frac{k}{2} \rceil$ as $f \to \infty$ and assuming that f is large enough, we can deduce that (2.37) holds if

$$\log\left(\frac{\lambda}{1+\lambda}\right) > -\frac{1}{r}\left(\left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - 1\right)\log f\right).$$

For this equation to hold, it suffices to show that we can find f and k such that $\lambda > \Gamma(f,k)$, where the function Γ is defined as

$$\Gamma(f,k) = \frac{\exp\Big(\Delta(f,k)\log\mu\Big)}{1-\exp\Big(\Delta(f,k)\log\mu\Big)},$$

$$\Delta(f,k) = \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - 1\right)\frac{\log f}{\log\Big(\mu^{-1}ck\mathbb{P}(\xi = k)\mathbb{P}(\mathcal{F} \geqslant f)\Big)}.$$

To complete the argument, it suffices to show that Assumption (**B**) implies that we can indeed choose f and k such that $\lambda > \Gamma(f,k)$. Let $C_1 \in [0,\infty)$ be such that

$$\limsup_{f\to\infty} \frac{\log \mathbb{P}(\mathcal{F} \geqslant f)}{\log f} = -C_1.$$

Then there exists a subsequence (f_n) such that $f_n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log \mathbb{P}(\mathcal{F} \geqslant f_n)}{\log f_n} = -C_1.$$

In particular, for any $\delta > 0$ for all n sufficiently large we have

$$\frac{\log \mathbb{P}(\mathcal{F} \geqslant f_n)}{\log f_n} \geqslant -C_1 - \delta.$$

Therefore, we obtain that

$$\limsup_{k\to\infty} \lim_{n\to\infty} \Delta(f_n, k) \leqslant \limsup_{k\to\infty} \frac{-1}{C_1+\delta}\left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - 1\right) = -\infty,$$

so that by choosing k and f_n large enough, we can guarantee that $\lambda > \Gamma(f_n, k)$ and the process survives strongly.

(ii) We now assume that condition (**C**) in Theorem 2.2.2 holds. This part follows in a similar manner to the proof of the previous case. As above, we can see that as $\mathbb{E}[\xi] < \infty$ we have that $k\mathbb{P}(\xi = k) \to 0$ and so $r = r(f, k) \to \infty$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \frac{1}{r} \log\left(\frac{32}{3}(2r+1)\right) = \lim_{r\to\infty} \frac{1}{r} \log\left(\frac{32}{3}(2r+1)\right) = 0.$$

Similarly to before we define

$$\begin{aligned}
\widetilde{\Gamma}(f,k) &= \frac{\exp\left(\widetilde{\Delta}(f,k)\log\mu\right)}{1-\exp\left(\widetilde{\Delta}(f,k)\log\mu\right)},\\
\widetilde{\Delta}(f,k) &= \left(\left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - 1\right)\log f - \log k\right)\frac{1}{\log\left(\mu^{-1}ck\mathbb{P}(\xi=k)\mathbb{P}(\mathcal{F}\geqslant f)\right)}.
\end{aligned}$$

Then, for large k and f, we have that if $\lambda>\widetilde{\Gamma}(f,k)$, then (2.37) holds and we can deduce the strong survival of the process. So our goal is to show that $\widetilde{\Delta}(f,k) \to -\infty$ as $k, f \to \infty$.

Let $C_2 \in [0,\infty)$ be such that

$$\limsup_{k\to\infty} \frac{\log \mathbb{P}(\xi = k)}{k} = -C_2.$$

Then there exists a sequence (k_n) such that $k_n \to \infty$ and $k_n^{-1} \log \mathbb{P}(\xi = k_n) \to -C_2$ as $n \to \infty$, so that for any $\delta > 0$, for n sufficiently large

$$k_n^{-1} \log \mathbb{P}(\xi = k_n) \geqslant -C_2 - \delta.$$

Now, since $\mu^{-1}c \leqslant 1$ we see that

$$\frac{\log k_n}{\log\left(\mu^{-1}ck_n\mathbb{P}(\xi = k_n)\mathbb{P}(\mathcal{F} \geqslant f)\right)} \geqslant 1.$$

Further, since $\mathbb{E}[\xi] < \infty$ we have $k_n\mathbb{P}(\xi = k_n) \to 0$ as $n \to \infty$ and therefore

$$\lim_{f\to\infty}\lim_{n\to\infty}\widetilde{\Delta}(f, k_n) \leqslant \lim_{f\to\infty} -\frac{\log f}{4(C_2+\delta)} - 1 = -\infty,$$

so that again for f and n sufficiently large $\lambda > \widetilde{\Gamma}(f, k_n)$ as required. This completes the proof. □

Part III

Continuous-state branching processes in a Lévy environment

Chapter 3

CSBPs in a Lévy environment

In this chapter, we introduce continuous-state branching processes (CSBP for short) in a Lévy random environment and discuss some of their main properties. The chapter is organised as follows. In Section 3.1, the CSBP in Lévy environment is defined as the strong solution of a stochastic differential equation (see [38, 58]). Further, in the case of finite mean we present a characterisation of its Laplace transform, given the environment, via a backward differential equation. In Section 3.3, some preliminaries of fluctuation theory of Lévy processes are introduced. Section 3.4 is devoted to define Lévy process conditioned to stay positive or negative, as well as some of their most useful properties. In Section 3.5 we introduce CSBPs in a conditioned Lévy environment whose properties are needed for our purposes in the next two chapters where we will study the asymptotic behaviour of the event of extinction and explosion. Finally, in Sections 3.6 and 3.7, we gather together some of the facts that are already in the literature regarding to extinction and explosion rates for CSBPs with stable branching mechanism in a Lévy environment.

3.1 Introduction

Continuous state branching processes in random environments (or CBPREs for short) are the continuous analogue, in time and space, of Galton-Watson processes in random environment (or GWREs for short). Roughly speaking, a process in this class is a strong Markov process taking values in $[0, \infty]$, where 0 and ∞ are absorbing states, satisfying the quenched branching property. Informally, the quenched branching property can be described as follows: conditionally on the environment, the process started from $x + y$ is distributed as the independent sum of two copies of the same process but issued from x and y, respectively.

CBPREs provides a richer class of branching models which take into account the effect of the environment on demographic parameters and let new phenomena appear. In particular, the classification of the asymptotic behaviour of rare events, such as the survival and explosion probabilities, is much more complex than the case when the environment is fixed since it may combine environmental and demographical stochasticities. Moreover, CBPREs also appear as scaling limits of GWREs which is a very rich family of population models; see for instance Kurtz [48] where the continuous path setting is considered and Bansaye and Simatos [8] and Bansaye et al. [7] where different classes of processes in random environment are studied including CBPREs.

An interesting family of CBPREs arises from several scalings of discrete population models in i.i.d. environments (see for instance [7, 8, 16]) which can be characterised by a stochastic differential equation whose linear term is driven by a Lévy process. Such Lévy process captures the effect of the environment on the mean offspring distribution of individuals. A process in this family is known as *continuous state branching process in Lévy environment* (or CSBP in Lévy environment for short) and its construction has been given by He et al. [38] and by Palau and Pardo [58], independently, as the unique strong solution of a stochastic differential equation as we will see below.

3.2 Definitions and first properties

Let $(\Omega^{(b)}, \mathcal{F}^{(b)}, (\mathcal{F}_t^{(b)})_{t\geqslant 0}, \mathbb{P}^{(b)})$ be a filtered probability space satisfying the usual hypothesis on which we may construct the branching term. We suppose that $(B_t^{(b)}, t \geqslant 0)$ is a $(\mathcal{F}_t^{(b)})_{t\geqslant 0}$-adapted standard Brownian motion, $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a $(\mathcal{F}_t^{(b)})_{t\geqslant 0}$-adapted Poisson random measure on $\mathbb{R}_+^3$, with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$ and μ satisfying

$$\int_{(0,\infty)} (1 \wedge z^2)\mu(\mathrm{d}z) < \infty.$$

The continuous-state branching process (CSBP for short) $(Y_t, t \geqslant 0)$ is defined as the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} Y_t = \quad Y_0 \quad &+ a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s^{(b)} \\ &+ \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Y_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \end{aligned}$$

where $a \in \mathbb{R}$ and $\gamma \geqslant 0$ (see Dawson and Li [24] for further details on strong existence and uniqueness). Further, every CSBP is characterised by the branching mechanism

ψ, a convex function of the form

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \Big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\Big)\mu(\mathrm{d}x), \qquad \lambda \geqslant 0. \tag{3.1}$$

On the other hand, for the environment term we consider another filtered probability space $(\Omega^{(e)}, \mathcal{F}^{(e)}, (\mathcal{F}_t^{(e)})_{t\geqslant 0}, \mathbb{P}^{(e)})$ satisfying the usual hypotheses. Let $\sigma \geqslant 0$ and α be real constants. Let π be a measure concentrated on $\mathbb{R}\setminus\{0\}$ such that

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

Suppose that $(B_t^{(e)}, t \geqslant 0)$ is a $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$ - adapted standard Brownian motion, $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$ - Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\pi(\mathrm{d}z)$, and $\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z)$ its compensated version. We denote by $(S_t, t \geqslant 0)$ the Lévy process with the following Lévy-Itô decomposition

$$S_t = \alpha t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z).$$

Note that, $(S_t, t \geqslant 0)$ is a Lévy process with no jumps smaller than -1 which is independent of the process $(Y_t, t \geqslant 0)$.

In our setting, the population size has no impact on the evolution of the environment and we are considering independent processes for the demography and the environment. More precisely, we work now on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ the direct product of the two probability spaces defined above, that is to say, $\Omega := \Omega^{(e)} \times \Omega^{(b)}, \mathcal{F} := \mathcal{F}^{(e)} \otimes \mathcal{F}^{(b)}, \mathcal{F}_t := \mathcal{F}_t^{(e)} \otimes \mathcal{F}_t^{(b)}$ for $t \geqslant 0$, $\mathbb{P} := \mathbb{P}^{(e)} \otimes \mathbb{P}^{(b)}$. Therefore, *the continuous-state branching process* $(Z_t, t \geqslant 0)$ *in a Lévy environment* $(S_t, t \geqslant 0)$ is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ as the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} Z_t = \ Z_0 \ + a\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s}\mathrm{d}B_s^{(b)} + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} zN^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s. \end{aligned}$$

According to [38, Theorem 3.1] or [57, Theorem 1], the equation has pathwise uniqueness and strong solution when $|\psi'(0+)| < \infty$. Moreover, Palau and Pardo [57] also consider the case when $\psi'(0+) = -\infty$, and obtained that the latter stochastic differential equation has a unique strong solution up to explosion and by convention here it is identically equal to ∞ after the explosion time. Furthermore, when conditioned on the environment, the process Z inherits the branching property of the underlying CSBP previously defined.

The analysis of the process Z is deeply related to the behaviour and fluctuations of the Lévy process $\xi = (\xi_t, t \geq 0)$, defined as follows

$$\xi_t = \overline{\alpha} t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s, \mathrm{d}z), \tag{3.2}$$

where

$$\overline{\alpha} := \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z).$$

Note that, both processes $(S_t, t \geqslant 0)$ and $(\xi_t, t \geqslant 0)$ generate the same filtration. Actually, the process ξ is obtained from S, changing only the drift and jump sizes.

The special case when we have finite mean, i.e., $|\psi'(0+)| < \infty$ or equivalently

$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) < \infty. \tag{3.3}$$

has already been extensively studied by Palau and Pardo in [58] and He et al in [38]. Let us therefore spend some time in this section gathering together some of the facts that the aforementioned authors already established in their papers. When we assume that the condition (3.3) holds, the auxiliary process can be taken as (3.2) but with a drift given as follows

$$\hat{\alpha} := \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z).$$

The process $\left(Z_t e^{-\xi_t}, t \geqslant 0\right)$ is a quenched martingale implying that for any $t \geqslant 0$ and $z \geqslant 0$,

$$\mathbb{E}_z[Z_t \mid S] = ze^{\xi_t}, \qquad \mathbb{P}\text{ -a.s,} \tag{3.4}$$

see Bansaye et al. [7]. In other words, the process ξ plays an analogous role as the random walk associated to the logarithm of the offspring means in the discrete time framework and leads to the usual classification for the long-term behaviour of branching processes. More precisely, we say that the process Z is subcritical, critical or supercritical accordingly as ξ drifts to $-\infty$, oscillates or drifts to $+\infty$.

Further, under the condition (3.3), there is another quenched martingale associated to $(Z_t e^{-\xi_t}, t \geqslant 0)$ which allow us to compute its Laplace transform, see for instance [58, Proposition 2] or [38, Theorem 3.4]. In order to compute the Laplace transform of $e^{-\xi_t} Z_t$, we first introduce the unique positive solution $(v_t(s, \lambda, \xi), s \in [0, t])$ of the following backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,\xi) = e^{\xi_s}\psi_0(v_t(s,\lambda,\xi)e^{-\xi_s}), \qquad v_t(t,\lambda,\xi) = \lambda \tag{3.5}$$

where

$$\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0+) = \gamma^2\lambda^2 + \int_{(0,\infty)} \Big(e^{-\lambda x} - 1 + \lambda x\Big)\mu(\mathrm{d}x). \tag{3.6}$$

Then the process $\Big(\exp\{-v_t(s,\lambda,\xi)Z_s e^{-\xi_s}\}, 0 \leq s \leq t\Big)$ is a quenched martingale implying that for any $\lambda \geqslant 0$ and $t \geqslant s \geqslant 0$,

$$\mathbb{E}\Big[\exp\{-\lambda Z_t e^{-\xi_t}\}\ \Big|\ S, \mathcal{F}_s^{(b)}\Big] = \exp\{-Z_s e^{-\xi_s} v_t(s,\lambda,\xi)\}. \tag{3.7}$$

Moreover, let us denote the random semigroup $h_{s,t}(\lambda) = e^{-\xi_s}v_t(s,\lambda e^{\xi_t},\xi)$ for all $\lambda \geqslant 0$ and $s \in [0,t]$. Thus,

$$\mathbb{E}\left[e^{-\lambda Z_t}\Big|\, S, \mathcal{F}_s^{(b)}\right] = \exp\{-Z_s h_{s,t}(\lambda)\}, \tag{3.8}$$

According to [38, Section 2], the mapping $s \mapsto h_{s,t}(\lambda)$ is the pathwise unique positive solution to the integral differential equation

$$h_{s,t}(\lambda) = e^{\xi_t - \xi_s}\lambda - \int_s^t e^{\xi_r - \xi_s}\psi_0\Big(h_{r,t}(\lambda)\Big)\mathrm{d}r, \qquad 0 \leqslant s \leqslant t. \tag{3.9}$$

We will obtain similar results in the case when we have $|\psi'(0+)| = \infty$. See Theorem 4.1.1 in the next Chapter.

3.3 Properties of the Lévy environment

In this section, we briefly recall the basic notations of Lévy processes and its fluctuation theory. For a more in-depth account of fluctuation theory, we refer the reader to the monographs of Bertoin [10], Doney [26] and Kyprianou [50].

Recall that $\xi = (\xi_t, t \geq 0)$ denotes the real valued Lévy process defined in (3.2). That is to say ξ has stationary and independent increments with càdlàg paths. For simplicity, we denote by $\mathbb{P}_x^{(e)}$ the law of the process ξ starting from $x \in \mathbb{R}$, i.e.

$$\mathbb{P}_x^{(e)}(\xi_t \in B) = \mathbb{P}^{(e)}(\xi_t + x \in B), \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}),$$

and when $x = 0$, we use the notation $\mathbb{P}^{(e)}$ for $\mathbb{P}^{(e)}_0$ (resp. $\mathbb{E}^{(e)}$ for $\mathbb{E}^{(e)}_0$). The dual process $\widehat{\xi} = -\xi$ is also a Lévy process satisfying that for any fixed time $t > 0$, the processes

$$(\xi_{(t-s)^-} - \xi_t, 0 \le s \le t) \qquad \text{and} \qquad (\widehat{\xi}_s, 0 \le s \le t), \tag{3.10}$$

have the same law, with the convention that $\xi_{0^-} = \xi_0$. For every $x \in \mathbb{R}$, let $\widehat{\mathbb{P}}^{(e)}_x$ be the law of $x + \xi$ under $\widehat{\mathbb{P}}^{(e)}$, that is the law of $\widehat{\xi}$ under $\mathbb{P}^{(e)}_{-x}$. In the sequel, we assume that ξ is not a compound Poisson process since it is possible that in this case the process visits the same maxima or minima at distinct times which can make our analysis more involved.

It is well-known that the law of the Lévy process ξ is determined by its characteristic exponent which is defined by $\Psi_\xi(\theta) := -\log \mathbb{E}^{(e)}[e^{i\theta\xi_1}]$, for $\theta \in \mathbb{R}$. Moreover, the characteristic exponent Ψ_ξ satisfies the so-called Lévy-Khintchine formula, i.e.

$$\Psi_\xi(\theta) = -\overline{\alpha} i\theta + \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} \Big(1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}}\Big)\pi(\mathrm{d}x).$$

Let us introduce the running infimum and supremum of ξ, by $\underline{\xi} = (\underline{\xi}_t, t \geqslant 0)$ and $\overline{\xi} = (\overline{\xi}_t, t \geqslant 0)$, with

$$\underline{\xi}_t = \inf_{0 \leqslant s \leqslant t} \xi_s \qquad \text{and} \qquad \overline{\xi}_t = \sup_{0 \leqslant s \leqslant t} \xi_s, \quad t \geqslant 0.$$

For our purposes, we also introduce and provide some useful properties of the Lévy process reflected at their running infimum and supremum. Let us recall that the reflected process $\xi - \underline{\xi}$ (resp. $\overline{\xi} - \xi$) is a Markov process with respect to the filtration $(\mathcal{F}^{(e)}_t)_{t \geqslant 0}$ and whose semigroup satisfies the Feller property (see for instance the monograph of Bertoin [10, Proposition VI.1]). We denote by $L = (L_t, t \geqslant 0)$ and $\widehat{L} = (\widehat{L}_t, t \geqslant 0)$ the local times of $\overline{\xi} - \xi$ and $\xi - \underline{\xi}$ at 0, respectively, in the sense of Chapter IV in [10]. If 0 is regular for $(-\infty, 0)$ or regular downwards, i.e.

$$\mathbb{P}^{(e)}(\tau_0^- = 0) = 1,$$

where $\tau_0^- = \inf\{s \ge 0 : \xi_s \le 0\}$, then 0 is regular for the reflected process $\xi - \underline{\xi}$ and then, up to a multiplicative constant, $\widehat{L}$ is the unique additive functional of the reflected process whose set of increasing points is $\{t : \xi_t = \underline{\xi}_t\}$. If 0 is not regular downwards then the set $\{t : \xi_t = \underline{\xi}_t\}$ is discrete and we define the local time $\widehat{L}$ as the counting process of this set. The same properties holds for L by duality, i.e. if 0 is regular upwards then, up to a multiplicative constant, L is the unique additive functional whose set of increasing points is $\{t : \xi_t = \overline{\xi}_t\}$, otherwise L is the counting process of this set.

Let us denote by L^{-1} and $\widehat{L}^{-1}$ the right continuous inverse of the local times L and $\widehat{L}$, respectively. The range of the inverse local times, L^{-1} (resp. $\widehat{L}^{-1}$), corresponds to the set of real times at which new maxima (resp. new minimum) occur. Next, define

$$H_t = \overline{\xi}_{L_t^{-1}}, \qquad t \geq 0. \tag{3.11}$$

The pair (L^{-1}, H) is a bivariate subordinator, as is the case of $(\widehat{L}^{-1}, \widehat{H})$ where

$$\widehat{H}_t = -\underline{\xi}_{\widehat{L}_t^{-1}}, \qquad t \geq 0.$$

The range of the process H (resp. $\widehat{H}$) corresponds to the set of new maxima (resp. new minimum). Both pairs are known as descending and ascending ladder processes, respectively. The Laplace transform of the ascending ladder process (L^{-1}, H) is such that for $\theta, \lambda \geqslant 0$,

$$\mathbb{E}^{(e)}\left[\exp\left\{-\theta L_t^{-1} - \lambda H_t\right\}\right] = \exp\left\{-t\kappa(\theta, \lambda)\right\}, \qquad t \geq 0, \tag{3.12}$$

writing $\kappa(\cdot, \cdot)$ for its bivariate Laplace exponent ($\widehat{\kappa}(\cdot, \cdot)$ for that of the descending ladder process) which, by an extension of the real-valued case, has the form

$$\kappa(\theta, \lambda) = \mathbf{a}\theta + \mathbf{b}\lambda + \int_{(0,\infty)^2} \left(1 - e^{-(\theta x + \lambda y)}\right) \Lambda(\,\mathrm{d}x,\,\mathrm{d}y),$$

where $\mathbf{a}$ and $\mathbf{b}$ are some non-negative constants representing the drift of L^{-1} and H, respectively and the bivariate measure $\Lambda(\mathrm{d}x, \mathrm{d}y)$ is concentrated in $(0, \infty)^2$ and satisfies

$$\int_{(0,\infty)^2} (x \wedge 1)(y \wedge 1) \Lambda(\,\mathrm{d}x,\,\mathrm{d}y) < \infty.$$

An interesting connection between the distributions of ξ and the ladder processes is given by the Wiener-Hopf factorisation

$$\mathbb{E}^{(e)}\left[e^{i\theta \xi_{\mathbf{e}_q}}\right] = \mathbb{E}^{(e)}\left[e^{i\theta \overline{\xi}_{\mathbf{e}_q}}\right] \mathbb{E}^{(e)}\left[e^{i\theta \underline{\xi}_{\mathbf{e}_q}}\right], \tag{3.13}$$

where $\mathbf{e}_q$ denotes an exponential random variable with parameter $q \geq 0$ which is independent of ξ. In addition, we have

$$\mathbb{E}^{(e)}\left[e^{i\theta \overline{\xi}_{\mathbf{e}_q}}\right] = \frac{\kappa(q, 0)}{\kappa(q, -i\theta)} \qquad \text{and} \qquad \mathbb{E}^{(e)}\left[e^{i\theta \underline{\xi}_{\mathbf{e}_q}}\right] = \frac{\widehat{\kappa}(q, 0)}{\widehat{\kappa}(q, i\theta)}.$$

We refer to Bertoin [11, Chapter VI] or Doney [26, Chapter 4] for further details on the ascending and descending ladder processes (H, L) and $(\widehat{H}, \widehat{L})$, respectively; as well as for the Wiener-Hopf factorisation.

Similarly to the critical case of the absorption rates studied by Bansaye et al. [7], the asymptotic analysis of rare events and the role of the initial condition involve the renewal functions U and $\widehat{U}$, associated to the supremum and infimum respectively, which are defined, for all $x > 0$, as follows

$$U(x) := \mathbb{E}^{(e)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\overline{\xi}_t \leqslant x\}} \,\mathrm{d}L_t\right] \quad \text{and} \quad \widehat{U}(x) := \mathbb{E}^{(e)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\underline{\xi}_t \geqslant -x\}} \,\mathrm{d}\widehat{L}_t\right]. \quad (3.14)$$

The renewal function U is finite, subadditive, continuous and increasing and moreover, they are identically 0 on $(-\infty, 0]$ and strictly positive on $(0, \infty)$. Also, U satisfies

$$U(x) \leqslant C_1 x, \qquad \text{for any} \qquad x \geqslant 0, \quad (3.15)$$

where C_1 is a finite constant (see for instance Lemma 6.4 and Section 8.2 in the monograph of Doney [26]). Moreover $U(0) = 0$ if 0 is regular upwards and $U(0) = 1$ otherwise. The same properties also holds for $\widehat{U}$.

Furthermore, it is important to note that by a simple change of variables we can relate the definitions of the renewal functions U and $\widehat{U}$ in terms of the ascending and descending ladder heights processes. Indeed, the measure induced by U and $\widehat{U}$ can be rewritten as follows,

$$U(x) = \mathbb{E}^{(e)}\left[\int_0^\infty \mathbf{1}_{\{H_t \leq x\}} \,\mathrm{d}t\right] \qquad \text{and} \qquad \widehat{U}(x) = \mathbb{E}^{(e)}\left[\int_0^\infty \mathbf{1}_{\{\widehat{H}_t \leq x\}} \,\mathrm{d}t\right].$$

Roughly speaking, the renewal function $U(x)$ (resp. $\widehat{U}(x)$) "measures" the amount of time that the ascending (resp. descending) ladder height process spends on the interval $[0, x]$ and in particular induces a measure on $[0, \infty)$ which is known as the renewal measure. The latter implies

$$\int_{[0,\infty)} e^{-\lambda x} U(\,\mathrm{d}x) = \int_0^\infty \mathbb{E}^{(e)}\left[e^{-\lambda H_t}\right] \,\mathrm{d}t = \frac{1}{\kappa(0, \lambda)}, \qquad \text{for} \quad \lambda \geq 0. \quad (3.16)$$

Similarly, we have

$$\int_{[0,\infty)} e^{-\lambda x} \widehat{U}(\,\mathrm{d}x) = \frac{1}{\widehat{\kappa}(0, \lambda)}, \qquad \text{for} \quad \lambda \geq 0. \quad (3.17)$$

We conclude this section with some remarks about moments and the exponential change of measure for Lévy processes. Recall that, there is a relationship between the moments of the Lévy measure and the moments of the distribution of the associated Lévy process at any fixed time.

In other words, the following condition

$$\text{there exists} \quad \vartheta^- \leqslant 0 < \vartheta^+ \text{ such that } \int_{\{|x|>1\}} e^{\theta x}\pi(\mathrm{d}x) < \infty, \quad \forall \theta \in [\vartheta^-, \vartheta^+] \tag{3.18}$$

is equivalent to the existence of the Laplace transform, i.e. $\mathbb{E}^{(e)}[e^{\theta\xi_1}]$ is well defined on $\theta \in [\vartheta^-, \vartheta^+]$ (see for instance Sato [68, Lemma 26.4]). The latter implies that we can introduce the Laplace exponent of ξ as follows $\Phi_\xi(\theta) = \log \mathbb{E}^{(e)}[e^{\theta\xi_1}]$, for $\theta \in [\vartheta^-, \vartheta^+]$, which clearly satisfies that $\Phi_\xi(\theta) = -\Psi_\xi(-i\theta)$. Again from [68, Lemma 26.4], we also have $\Phi_\xi(\theta) \in C^\infty$ and $\Phi''_\xi(\theta) > 0$, for $\theta \in (\vartheta^-, \vartheta^+)$.

Another object which will be relevant for our analysis in the next chapters is the so-called exponential martingale associated to the Lévy process ξ, i.e.

$$M_t^{(\theta)} = \exp\Big\{\theta\xi_t - t\Phi_\xi(\theta)\Big\}, \qquad t \geq 0,$$

which is well-defined for $\theta \in [\vartheta^-, \vartheta^+]$ under assumption (3.18). It is well-known that $(M_t^{(\theta)}, t \geq 0)$ is a $(\mathcal{F}_t^{(e)})_{t\geq 0}$-martingale and that it induces a change of measure which is known as the Esscher transform, that is to say

$$\mathbb{P}^{(e,\theta)}(\Lambda) = \mathbb{E}^{(e)}\Big[M_t^{(\theta)}\mathbf{1}_\Lambda\Big], \qquad \text{for} \quad \Lambda \in \mathcal{F}_t^{(e)}. \tag{3.19}$$

This change of measure has the important property that the process ξ under $\mathbb{P}^{(e,\theta)}$ is still a Lévy process (see for instance Kyprianou [50, Theorem 3.9]). Similarly as above, we introduce the corresponding renewal functions under this change of measure $\mathbb{P}^{(e,\theta)}$, i.e. for $x \geq 0$,

$$U^{(\theta)}(x) := \mathbb{E}^{(e,\theta)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\overline{\xi}_t \leqslant x\}}\,\mathrm{d}L_t\right] \quad \text{and} \quad \widehat{U}^{(\theta)}(x) := \mathbb{E}^{(e,\theta)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\underline{\xi}_t \geqslant -x\}}\,\mathrm{d}\widehat{L}_t\right]. \tag{3.20}$$

3.4 Lévy processes conditioned to stay positive and negative

Lévy processes conditioned to stay positive are well studied objects. For a complete overview of this theory the reader is referred to [10, 21, 22] and references therein. Nevertheless, it is worth spending a little time here investigating the type of conditioning we are interested in the following chapters.

Let us define the probability $\mathbb{Q}_x$ associated to the Lévy process ξ started at $x > 0$ and killed at time ζ when it first enters $(-\infty, 0)$, that is to say

$$\mathbb{Q}_x\Big[f(\xi_t)\mathbf{1}_{\{\zeta>t\}}\Big] := \mathbb{E}_x^{(e)}\Big[f(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\Big],$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is measurable. It is important to note that $\mathbb{Q}_0$ is well defined when 0 is not regular downwards.

According to Chaumont and Doney [22, Lemma 1], under the assumption that ξ does not drift towards $-\infty$, we have that the renewal function $\widehat{U}$ is invariant for the killed process. In other words, for all $x > 0$ and $t \geq 0$,

$$\mathbb{Q}_x\left[\widehat{U}(\xi_t)\mathbf{1}_{\{\zeta>t\}}\right] = \mathbb{E}_x^{(e)}\left[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\right] = \widehat{U}(x). \tag{3.21}$$

Hence, from the Markov property, we deduce that $\{\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}, t \geqslant 0\}$ is a martingale with respect to $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$. We may now use this martingale to define a change of measure corresponding to the law of ξ *conditioned to stay positive* as a Doob-h transform. Before doing so, let us recall that ξ is adapted to the filtration $(\mathcal{F}_t^{(e)})_{t\geq 0}$. Under the assumption that ξ does not drift towards $-\infty$, the law of the process ξ conditioned to stay positive is defined as follows, for $\Lambda \in \mathcal{F}_t^{(e)}$ and $x > 0$,

$$\mathbb{P}_x^{(e),\uparrow}(\Lambda) := \frac{1}{\widehat{U}(x)}\mathbb{E}_x^{(e)}\left[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\mathbf{1}_\Lambda\right]. \tag{3.22}$$

When 0 is not regular downwards, the above definition still makes sense for $x = 0$.

The measure $\mathbb{P}_x^{(e),\uparrow}$, corresponds to the limit as ϵ goes to 0 of the law of the process conditioned to stay positive up to an independent exponential time with parameter ϵ (see Chaumont [21, Theorem 1]).

Lemma 3.4.1 (Chaumont [21])**.** *Let* $\mathbf{e}_1$ *be an exponential random variable with parameter 1 which is independent of* ξ. *For any* $x > 0$, *any* $(\mathcal{F}_t^{(e)})_{t\geq 0}$ *stopping time* τ *and* $\Lambda \in \mathcal{F}_\tau^{(e)}$, *we have*

$$\lim_{\epsilon\to 0}\mathbb{P}_x^{(e)}\left(\Lambda, \tau < \mathbf{e}_1/\epsilon \,\middle|\, \xi_u > 0, 0 \leqslant u \leqslant \mathbf{e}_1/\epsilon\right) = \mathbb{P}_x^{(e),\uparrow}(\Lambda). \tag{3.23}$$

Similarly, by duality, under the assumption that ξ does not drift towards ∞, the law of the process ξ *conditioned to stay negative* is defined for $x < 0$, as follows

$$\mathbb{P}_x^{(e),\downarrow}(\Lambda) := \frac{1}{U(-x)}\widehat{\mathbb{E}}_{-x}^{(e)}\left[U(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\mathbf{1}_\Lambda\right]. \tag{3.24}$$

Let us finish this section by presenting some remarks in the case when the Lévy process drifts to $-\infty$, i.e. $\lim_{t\to\infty}\xi_t = -\infty$. In this case, the equality in (3.21) is replaced by the following inequality

$$\mathbb{E}_x^{(e)}\left[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\right] \leqslant \widehat{U}(x),$$

and we have that $\{\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}, t \geqslant 0\}$ is a supermartingale with respect to $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$. Moreover, the change of measure (3.22) must take place on the space of processes which are killed at some time and sent to a cemetery state (see for instance Kyprianou [50, Section 13.2.1] for further details).

Hirano [39] investigated the asymptotic behaviour of a Lévy process with negative drift conditioned to stay positive using a conditioning similar to that given in the left-hand side of (3.23). More precisely, Hirano proved the following result.

Theorema 3.4.2 (Hirano [39]). *Assume that condition* (3.18) *holds with* $\vartheta^- = 0$ *and that there exists* $\gamma \in (0, \vartheta^+)$ *such that* $\Phi'_\xi(\gamma) = 0$. *Let* $x > 0$, *thus for* $s \geqslant 0$ *and* $\Lambda \in \mathcal{F}_s^{(e)}$, *we have*

$$\lim_{t\to\infty}\mathbb{P}_x^{(e)}\Big(\xi \in \Lambda \mid \underline{\xi}_t > 0\Big) = \mathbb{P}_x^{(e,\gamma),\uparrow}(\xi \in \Lambda).$$

Note that the measure $\mathbb{P}_x^{(e,\gamma),\uparrow}$ corresponds to the process conditioned to stay positive after making a change of measure using the Esscher transform given in (3.19). In other words, this approximation leads to the law of a certain oscillating Lévy process conditioned to stay positive. The proof of Theorem 3.4.2 is mainly based on the following resul.

Theorema 3.4.3 (Hirano [39]). *Assume that condition* (3.18) *holds with* $\vartheta^- = 0$ *and also suppose that* $\Phi'_\xi(0) = 0$. *Let* $x > 0$ *and* $\theta > 0$, *then,*

$$\lim_{t\to\infty} t^{3/2}\mathbb{E}_x^{(e)}\left[e^{-\theta\xi_t}\mathbf{1}_{\{\underline{\xi}_t>0\}}\right] = \frac{C_0}{\sqrt{2\pi\Phi''_\xi(0)}}\widehat{U}(x)\int_0^\infty e^{-\theta y}U(y)\mathrm{d}y,$$

where

$$C_0 := \exp\left\{\int_0^\infty \frac{(e^{-t}-1)}{t}\mathbb{P}^{(e)}(\xi_t = 0)\mathrm{d}t\right\}.$$

3.5 CSBPs in a conditioned Lévy environment

Similarly to the definition of Lévy processes conditioned to stay positive given above and following a similar strategy as in the discrete framework in Afanasyev et al. [3], we would like to introduce a CSBP in a Lévy environment conditioned to stay positive as a Doob-h transform. The aforementioned CSBP process was first investigated by Bansaye et al. [7] with the aim to study the survival event in a critical Lévy environment. In other words, they proved the following result.

Lemma 3.5.1 (Bansaye et. al. [7])**.** *Let us assume that $z, x > 0$. The process $\{\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t > 0\}}, t \geqslant 0\}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geqslant 0}$ and under $\mathbb{P}_{(z,x)}$.*

With this in hand, they introduce the law of a *CSBP in a Lévy environment ξ conditioned to stay positive* as follows, for $\Lambda \in \mathcal{F}_t$, $z, x > 0$,

$$\mathbb{P}^{\uparrow}_{(z,x)}(\Lambda) := \frac{1}{\widehat{U}(x)}\mathbb{E}_{(z,x)}\Big[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t > 0\}}\mathbf{1}_{\Lambda}\Big],$$

where $\widehat{U}$ is the renewal function defined in (3.14). It is natural therefore to cast an eye on similar issues for the study of non-explosion events in the critical Lévy environment (see Chapter 4 below). In contrast, we introduce here the process Z in a Lévy environment ξ conditioned to stay negative. Recall that $\widehat{\xi}$ is the dual process of ξ. Appealing to the duality and Lemma 3.5.1, we can see that the process $\{U(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t < 0\}}, t \geqslant 0\}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geqslant 0}$ and under $\mathbb{P}_{(z,x)}$ with $z > 0$ and $x < 0$. Then we introduce the law of the *CSBP in a Lévy environment ξ conditioned to stay negative*, as follows: for $\Lambda \in \mathcal{F}_t$ for $z > 0$ and $x < 0$,

$$\mathbb{P}^{\downarrow}_{(z,x)}(\Lambda) := \frac{1}{U(-x)}\mathbb{E}_{(z,x)}\Big[U(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t < 0\}}\mathbf{1}_{\Lambda}\Big]. \tag{3.25}$$

Intuitively speaking, $\mathbb{P}^{\uparrow}_{(z,x)}$ and $\mathbb{P}^{\downarrow}_{(z,x)}$ correspond to the law of (Z, ξ) conditioning the random environment ξ not to enter $(-\infty, 0)$ and $(0, \infty)$, respectively.

3.6 Extinction for stable CSBPs in a Lévy environment

One of our aims is to study the asymptotic behaviour of the non-extinction probability for CSBPs in a Lévy environment. A flavour for this has already been given by the authors in [52, 59]. The purpose of this section is to present some of the known results in the literature on the extinction event for the aforementioned family of processes.

We begin by assuming that condition (3.3) holds, which ensures non-explosivity (see Section 3.7 below for further details). In addition, to focus on the absorption event, we use *Grey's condition* which guarantees that Z hits zero with strictly positive probability. More precisely, we say that Z satisfies Grey's condition if

$$\int_1^\infty \frac{\mathrm{d}z}{\psi_0(z)} < \infty, \tag{3.26}$$

where $\psi_0(\lambda)$ is as in (3.6). Recently, He et al. [38] have shown that this condition is necessary and sufficient for a CSBP in a Lévy environment to get absorbed with positive probability (see [38, Theorem 4.1]), i.e.,

$$\mathbb{P}_z(Z_t = 0) > 0 \quad \text{for all} \quad t > 0.$$

When the branching mechanism is stable, we can obtain in an explicit way the absorption event in terms of an exponential functional of the Lévy process ξ. More precisely, denote by ψ the stable branching mechanism, i.e., for $\lambda \geqslant 0$

$$\psi(\lambda) = C\lambda^{1+\beta}, \tag{3.27}$$

where $\beta \in (-1, 0) \cup (0, 1)$ and C is a constant such that: $C < 0$ if $\beta \in (-1, 0)$ and $C > 0$ if $\beta \in (0, 1)$. According to [58], the non-extinction probability for a stable CSBP in a Lévy environment ξ is given by

$$\mathbb{P}_z(Z_t > 0) = 1 - \mathbb{E}^{(e)}\left[\exp\left\{-z\Big(\beta C \mathtt{I}_{0,t}(\beta\xi)\Big)^{-1/\beta}\right\}\right]\mathbf{1}_{\{\beta>0\}}, \qquad z \geqslant 0, \tag{3.28}$$

where we recall that $\mathbb{E}^{(e)}$ denotes the expectation under $\mathbb{P}^{(e)}$, which corresponds to the law of ξ starting from $x = 0$, and $\mathtt{I}_{0,t}(\beta\xi)$ denotes the exponential functional of the Lévy process $\beta\xi$, i.e.,

$$\mathtt{I}_{0,t}(\beta\xi) := \int_0^t e^{-\beta\xi_s}\mathrm{d}s, \qquad t \geqslant 0. \tag{3.29}$$

From (3.28), we see that for $\beta \in (-1, 0)$ the process Z survives $\mathbb{P}_z$-a.s. Let us focus on the most interesting case, i.e. $\beta \in (0, 1)$. Palau and Pardo in [57] studied the asymptotic behaviour of the survival probability of stable CSBP in a Browian environment. Afterwards, Palau and co-authors in [59] extended this result to the case when the environment is driven by a general Lévy process. We state this result here for the sake of completeness.

Theorema 3.6.1 (Palau et al. [59])**.** *Suppose that condition* (3.18) *holds with* $\vartheta^- = 0$ *and* $\vartheta^+ > 1$*. Let* $(Z_t, t \geqslant 0)$ *be the stable CSBP with index* $\beta \in (0,1)$*,* $Z_0 = z > 0$ *and in a Lévy environment.*

1. *Supercritical regime. If* $\Phi'_\xi(0+) > 0$*, then*
$$\lim_{t\to\infty} \mathbb{P}_z(Z_t > 0) = \mathbb{E}^{(e)}\Big[1 - \exp\Big\{-z\Big(\beta c \mathrm{I}_{0,\infty}(\beta\xi)\Big)^{-1/\beta}\Big\}\Big] > 0.$$

2. *Critical regime. If* $\Phi'_\xi(0+) = 0$*, then for each* $z > 0$*, there exists* $c_1(z) > 0$ *such that*
$$\lim_{t\to\infty} t^{1/2}\mathbb{P}_z(Z_t > 0) = c_1(z).$$

3. *Subcritical regime. Assume that* $\Phi'_\xi(0+) < 0$*, then*

   (a) *Strongly subcritical regime. If* $\Phi'_\xi(1) < 0$*, then there exists* $c_2 > 0$ *such that for every* $z > 0$*,*
   $$\lim_{t\to\infty} e^{-t\Phi_\xi(1)}\mathbb{P}_z(Z_t > 0) = c_2 z.$$

   (b) *Intermediate subcritical regime. If* $\Phi'_\xi(1) = 0$*, then there exists* $c_3 > 0$ *such that for every* $z > 0$*,*
   $$\lim_{t\to\infty} t^{1/2} e^{-t\Phi_\xi(1)}\mathbb{P}_z(Z_t > 0) = c_3 z.$$

   (c) *Weakly subcritical. If* $\Phi'_\xi(1) > 0$*, then for each* $z > 0$*, there exists* $c_4(z) > 0$ *such that*
   $$\lim_{t\to\infty} t^{3/2} e^{-t\Phi_\xi(\gamma)}\mathbb{P}_z(Z_t > 0) = c_4(z).$$
   where $\gamma \in (0,1)$ *satisfies* $\Phi'_\xi(\gamma) = 0$*.*

A similar result was also obtained by Li and Xu [52] independently, where the limiting coefficients were given explicitly (see [52, Teorema 2.8]). More recently, Bansaye et al. [7] studied the survival probability for a larger class of branching mechanisms associated to CSBPs in Lévy environments. They focus on the critical case, more precisely in oscillating Lévy environments satisfying the so-called *Spitzer's condition* at ∞. This condition states that the expected proportion of time at which the Lévy process spends within the positive real half line up to time t, stabilizes as $t \to \infty$ at some value between 0 and 1. That is to say,

$$\frac{1}{t}\int_0^t \mathbb{P}^{(e)}(\xi_s \geqslant 0)\mathrm{d}s \longrightarrow \rho \in (0,1), \qquad \text{as} \qquad t \to \infty. \tag{\textbf{H1}}$$

Moreover, Bertoin and Doney in [11] showed that the later condition is equivalent to $\mathbb{P}^{(e)}(\xi_s \geqslant 0) \to \rho$, as $s \to \infty$. Under Spitzer's condition (see Bertoin [10, Theorem VI.18]) the asymptotic behaviour of the probability that the Lévy process ξ remains positive, i.e. $\mathbb{P}^{(e)}_x(\underline{\xi}_t > 0)$ for $x > 0$, is regularly varying at ∞ with index $\rho - 1$ and moreover, for any $x, y > 0$, we have

$$\lim_{t\to\infty} \frac{\mathbb{P}^{(e)}_x\Big(\underline{\xi}_t > 0\Big)}{\mathbb{P}^{(e)}_y\Big(\underline{\xi}_t > 0\Big)} = \frac{\widehat{U}(x)}{\widehat{U}(y)}. \tag{3.30}$$

In other words, we obtain that for any $x > 0$,

$$\mathbb{P}^{(e)}_x\Big(\underline{\xi}_t > 0\Big) \sim \widehat{U}(x)t^{\rho-1}\ell(t), \quad \text{as} \quad t \to \infty, \tag{3.31}$$

where ℓ is a slowly varying function at ∞, that is to say, for all $c > 0$,

$$\lim_{t\to\infty} \frac{\ell(ct)}{\ell(t)} = 1.$$

Moreover, Bansaye et al. [7] also assume that the Lévy measure μ fulfils the following $x\log^2 x$ moment condition

$$\int^{\infty} x\log^2 x\mu(\,\mathrm{d}x) < \infty. \tag{\textbf{H2}}$$

This moment assumption allows them to guarantee the non-extinction of the process in "favorable" environments, or in other words when the running infimum of the Lévy environment is positive. In addition, for the branching mechanism they make a slightly stronger assumption:

$$\text{there exists } \beta \in (0,1] \text{ and } C > 0 \text{ such that } \psi_0(\lambda) \geqslant C\lambda^{1+\beta} \quad \text{for } \lambda \geqslant 0. \tag{\textbf{H3}}$$

In particular, note that the latter assumption guarantees that ψ_0 satisfies the Grey's condition (3.26). Let us finish this section by presenting their result.

Theorema 3.6.2 (Bansaye et al. [7]). *Let $(Z_t, t \geqslant 0)$ the CSBP in a Lévy environment. Assume that conditions* (**H1**)-(**H3**) *hold, then there exists a positive function c* *such that for any $z > 0$,*

$$\mathbb{P}_z(Z_t > 0) \sim c(z)\mathbb{P}_1^{(e)}\Big(\underline{\xi}_t > 0\Big) \sim c(z)\widehat{U}(1)t^{\rho-1}\ell(t), \quad \textit{as} \quad t \to \infty,$$

where ℓ is the slowly varying function defined in (3.31)*.*

3.7 Explosion for CSBPs in a Lévy environment

In the case of the event of explosion for CSBPs in a general Lévy environment, there are a few results about it. This is also the case for the speed of the probability of non-explosion. In this section, we gather together the results that are already known in the literature. In Chapter 4, we extend these results in a more general setting.

We say that Z is a *conservative* process or, in other words, that there is no explosion in finite time if

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \text{for all} \quad t > 0, \tag{3.32}$$

and any $z \geqslant 0$. In the case of a CSBP with constant environment, Grey in [35] provided necessary and sufficient conditions for the process to be conservative which depend on the integrability of the mapping $z \mapsto (|\psi(z)|)^{-1}$ near 0. That is a CSBP is conservative if and only if

$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \infty.$$

Observe that, a necessary condition is that $\psi(0) = 0$ and a sufficient condition is that $\psi(0) = 0$ and $|\psi'(0+)| < \infty$ (see for instance Kyprianou [50, Theorem 12.3]).

In contrast, in the particular case when the environment is driving by a Brownian motion, Palau and Pardo [58] furnish only a necessary condition under which the process Z is conservative. They proved that if the branching mechanism satisfies $|\psi'(0+)| < \infty$, then the CSBP in a Brownian environment is conservative (see [57, Proposition 1]). More recently, Bansaye et al. in [7] extended this result in the context when the environment is driven by a Lévy process. They showed that under the condition $|\psi'(0+)| < \infty$ or equivalently (3.3) holds, the process Z is conservative (see [7, Lemma 7]).

The asymptotic behaviour of the non-explosion probability and the explosion event have not been studied in a general form. Up to our knowledge, the long-term behaviour has been only studied for the case where the associated branching mechanism is stable since the non-explosion probability can be written explicitly in terms of the exponential functional of ξ. Recall that the stable branching mechanism ψ satisfies

(3.27). According to [58], the non-explosion probability for a stable CSBP in a Lévy environment ξ is given by

$$\mathbb{P}_z(Z_t < \infty) = \mathbf{1}_{\{\beta>0\}} + \mathbf{1}_{\{\beta<0\}}\mathbb{E}^{(e)}\left[\exp\left\{-z\Big(\beta C\mathtt{I}_{0,t}(\beta\xi)\Big)^{-1/\beta}\right\}\right], \qquad z \geqslant 0, \quad (3.33)$$

where we recall that $C<0$ if $\beta \in (-1,0)$ and $C>0$ if $\beta \in (0,1)$, and also that $\mathtt{I}_{0,t}(\beta\xi)$ denotes the exponential functional of the Lévy process $\beta\xi$, as it is defined in (3.29). Note that, from identity (3.33), we know that there is a positive probability that the process explodes for any $t>0$ and $z>0$. Furthermore, Palau et al. [59, Proposition 2.1] found three different regimes for the asymptotic behaviour of the non-explosion probability that depends on the mean of the random environment. They called these regimes: *subcritical-explosion, critical-explosion* and *supercritical-explosion* depending on whether the Lévy environment ξ drifts to $-\infty$, oscillates or drifts to ∞. More precisely, they proved the following result.

Theorema 3.7.1 (Palau et al. [59])**.** *Suppose that condition* (3.18) *holds with* $\vartheta^- < 0 < \vartheta^+$. *Let* $(Z_t, t \geqslant 0)$ *be the stable CSBP with index* $\beta \in (-1,0)$, $Z_0 = z > 0$ *and in a Lévy environment.*

1. *Subcritical-explosion. If* $\Phi'_\xi(0+) < 0$, *then*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t<\infty) = \mathbb{E}^{(e)}\left[\exp\left\{-z\Big(\beta c\mathtt{I}_{0,\infty}(\beta\xi)\Big)^{-1/\beta}\right\}\right] > 0.$$

2. *Critical-explosion. If* $\Phi'_\xi(0+) = 0$, *then for each* $z>0$, *there exists* $c_1(z)>0$ *such that*

$$\lim_{t\to\infty} t^{1/2}\mathbb{P}_z(Z_t<\infty) = c_1(z).$$

3. *Supercritical-explosion. Assume that* $\Phi'_\xi(0+) > 0$, *then for each* $z>0$, *there exists* $c_2(z)>0$ *such that*

$$\lim_{t\to\infty} t^{3/2}e^{-t\Phi_\xi(\gamma)}\mathbb{P}_z(Z_t<\infty) = c_2(z),$$

   where $\gamma \in (\vartheta^-, 0)$ *is such that* $\Phi'_\xi(\gamma) = 0$.

Up to our knowledge, this is the only known result in the literature about explosion rates for CSBPs in a Lévy environment. As we said before, in Chapter 4, we study the explosion rates for CSBPs in a Lévy environment for a more general setting.

Chapter 4

Explosion rates of CSBPs in a Lévy environment

Here, we study the speed of the probability of non-explosion for continuous state branching processes in a Lévy environment, where the associated Lévy process either oscillates or drifts to $-\infty$ and the branching mechanism is given by the negative of the Laplace exponent of a subordinator. Assuming that the Lévy process associated to the environment either satisfies Spitzer's condition and the existence of some moments on its associated descending ladder height or drifts to $-\infty$ and under an integrability condition, we extend recent results in the case where the branching mechanism is assumed to be stable. In order to do so, we require to study the law of the CSBP in Lévy environment in the non-finite mean case and furnishes necessary and sufficient conditions for the process to be conservative, i.e. that the process does not explode a.s. The chapter is structured as follows. In Section 4.1, we present our main results. In Section 4.2, we study the law of the process in the case of non-finite mean. Section 4.3 is devoted to some results about the CSBPs in a conditioned Lévy environment which was introduced in Chapter 3. The aim of Section 4.4 is to present the proof of the asymptotic behaviour of the non-explosion probability in the case of a critical Lévy environment. In Section 4.5, we study the speed of the non-explosion probability in a Lévy environment drifting to $-\infty$. Finally, in Section 4.6 we state some conjectures about the explosion problem for CSBPs in a Lévy environment drifting to $+\infty$.

4.1 Main results

Let $Z = (Z_t, t \geqslant 0)$ be the continuous-state branching process in a Lévy environment $(S_t, t \geqslant 0)$ defined in Chapter 3. We begin by recalling that the branching mechanism

ψ is either the negative of the Laplace exponent of a subordinator or the Laplace exponent of a spectrally negative Lévy process. Here we focus on the subordinator case. In other words, we shall assume that $\psi(\lambda) = -\phi(\lambda)$, where ϕ is a concave, increasing and non-negative function satisfying

$$\phi(\lambda) = \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\mu(\mathrm{d}x), \qquad \lambda \geqslant 0, \tag{4.1}$$

with

$$\delta := a - \int_{(0,1)} x\mu(\mathrm{d}x) \geqslant 0 \qquad \text{and} \qquad \int_{(0,\infty)} (1 \wedge x)\mu(\mathrm{d}x) < \infty.$$

Alternatively, using integration by parts, the function ϕ can also be rewritten in terms of the Laplace transform of the tail of μ, that is to say,

$$\phi(\lambda) = \delta\lambda + \lambda \int_{(0,\infty)} e^{-\lambda x}\bar{\mu}(x)\mathrm{d}x, \tag{4.2}$$

where $\bar{\mu}(x) := \mu(x, \infty)$.

In what follows, we may assume, without loss of generality, that the drift of the branching mechanism is zero, i.e. $\delta = 0$. Otherwise, we consider the branching mechanism

$$\psi(\lambda) = -(\phi(\lambda) - \delta\lambda) = -\lambda \int_{(0,\infty)} e^{-\lambda x}\bar{\mu}(x)\mathrm{d}x,$$

and modify the Lévy process $\xi = (\xi_t, t \geq 0)$ by adding δ to the drift, that is to say

$$\xi_t = \alpha_1 t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s, \mathrm{d}z),$$

where

$$\alpha_1 := \alpha - \delta - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z). \tag{4.3}$$

In other words, it is enough to consider a subordinator with no drift since the latter can be included in the environment. This procedure will turn out to be important in the proof of Lemma 4.3.3 below.

As we mentioned earlier in Section 3.2, in the case when we have finite mean, i.e. $|\psi'(0+)| < \infty$, the branching mechanism ψ determines the law of the reweighted process $(Z_t e^{-\xi_t}, t \geqslant 0)$ via a backward ordinary differential equation (the reader is again referred to [58, Proposition 2] and [38, Theorem 3.4]). Here, we obtain a similar result in the infinite mean case but when the branching mechanism is given by the negative of a subordinator without drift. In our proof, we show existence of the solution of the aforementioned backward differential equation, uniqueness seems to be difficult to be deduced since in this case ψ is not globally Lipschitz. More precisely, we have the following result.

Theorema 4.1.1. *Assume that $\psi(\lambda) = -\phi(\lambda)$ with $\phi'(0+) = \infty$. For every $z, \lambda, t > 0$ and $x \in \mathbb{R}$, we have*

$$\mathbb{E}_{(z,x)}\Big[\exp\Big\{-\lambda Z_t e^{-\xi_t}\Big\}\Big|\,\xi\Big] = \exp\Big\{-z v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0)\Big\}, \tag{4.4}$$

where for any $\lambda, t \geqslant 0$, the function $v_t : s \in [0,t] \to v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0)$ is an a.s. solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,\xi) = e^{\xi_s}\psi\Big(v_t(s,\lambda,\xi)e^{-\xi_s}\Big), \quad a.e. \quad s \in [0,t] \tag{4.5}$$

and with terminal condition $v_t(t,\lambda,\xi) = \lambda$.

Recall from Chapter 3 that Z is a *conservative* process if there is no explosion in finite time a.s., i.e. (3.32) holds. Our second main result furnishes a necessary and sufficient condition for a CSBP in a Lévy environment to be conservative. The result is an extension of the original characterisation given by Grey [35] in the classical case for CSBP with constant environment.

Proposition 4.1.2. *Assume that $\psi(\lambda) = -\phi(\lambda)$. A continuous-state branching process in a Lévy environment with branching mechanism ψ is conservative if and only if*

$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \infty. \tag{4.6}$$

We deal now with the asymptotic behaviour of the non-explosion probability in the critical and subcritical regimes. First we focus on the *critical-explosion* regime. More precisely, we are assuming that the Lévy environment satisfies the so-called *Spitzer's condition* at ∞, i.e. that assumption (**H1**) holds. Recall from (3.12) that $\kappa(\theta,\lambda)$ is the Laplace exponent of the ascending ladder process (L^{-1}, H). Now, according to Bertoin [10, Theorem 12, Chapter IV], the Spitzer's condition (**H1**) is equivalent to the Laplace exponent $\kappa(\cdot, 0)$ of the ascending ladder time process L^{-1} being regularly varying at 0^+ with index $\rho \in (0,1)$. To be more precise, for $\rho \in (0,1)$ we have,

$$\lim_{t\downarrow 0} \frac{\kappa(ct,0)}{\kappa(t,0)} = c^\rho, \qquad \text{for all} \qquad c > 0.$$

In addition, the function $\kappa(\cdot,\ 0)$ may always be written in the form

$$\kappa(t,0) = t^\rho \ell_1(t), \tag{4.7}$$

where ℓ_1 is a slowly varying function at 0^+. That is to say, for all positive constant c, the function ℓ_1 satisfies

$$\lim_{x\downarrow 0}\frac{\ell_1(cx)}{\ell_1(x)}=1.$$

Denote by τ_x^+ and τ_x^-, $x\in\mathbb{R}$, the first passage times through a barrier at level x for the Lévy process ξ, i.e.,

$$\tau_x^+ := \inf\{t\geqslant 0:\xi_t\geqslant x\}\qquad\text{and}\qquad \tau_x^- := \inf\{t\geqslant 0:\xi_t\leqslant x\},\qquad x\in\mathbb{R},$$

with the usual assumption that $\inf\emptyset=\infty$. Now we remark that, by [49], we may deduce the asymptotic behaviour of the probability that the Lévy process ξ remains negative. In other words, under the Spitzer's condition (**H1**) or equivalently under the assumption that $\kappa(\cdot,0)$ is regularly varying at zero with index $\rho\in(0,1)$, we have the asymptotic behaviour of $\mathbb{P}^{(e)}_x(\overline{\xi}_t<0)=\mathbb{P}^{(e)}(\tau^+_{-x}>t)$ for $x<0$. That is to say,

$$\lim_{t\to\infty}\frac{\sqrt{\pi}}{\kappa(1/t,0)}\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t<0\Big)=\lim_{t\to\infty}\frac{\sqrt{\pi}}{\kappa(1/t,0)}\widehat{\mathbb{P}}^{(e)}_{-x}\Big(\underline{\xi}_t<0\Big)=U(-x),\tag{4.8}$$

where we recall that $U(\cdot)$ is the renewal function for the ascending ladder-height process defined in (3.14) (see also Bertoin [10, Theorem VI.18]).

In order to control the effect of the environment on the event of non-explosion we need others assumptions. The following moment condition on the descending ladder height process associated to ξ is needed to guarantee the non-explosion of the process in unfavorable environments. Let us assume

$$\widehat{\mathbb{E}}^{(e)}\Big[H_1e^{H_1}\Big]<\infty,\tag{\textbf{A1}}$$

where we recall that $\widehat{\mathbb{E}}^{(e)}$ denotes the expectation associated to the law $\widehat{\mathbb{P}}^{(e)}$ of the dual process $\hat{\xi}=-\xi$ and H denotes its associated ascending ladder height.

On the other hand, we shall assume that the branching mechanism is upper bounded by a stable branching mechanism whose associated CSBP in a Lévy environment explodes with positive probability. More precisely, we assume that

$$\text{there exists }\beta\in(-1,0)\text{ and }C<0\text{ such that }\ \psi(\lambda)\leqslant C\lambda^{1+\beta}\quad\text{ for all }\lambda\geqslant 0.\tag{\textbf{A2}}$$

Observe that, under condition (**A2**) and using the representation of ϕ given in (4.2), we deduce that for all $\lambda > 0$

$$\int_{(0,\infty)} e^{-\lambda x}\bar{\mu}(x)\mathrm{d}x \geqslant -C\lambda^{\beta}, \tag{4.9}$$

where we recall that $\delta = 0$ in accordance with the observation above Theorem 4.1.1. Now, letting $\lambda \downarrow 0$ in the previous inequality, we see that condition (**A2**) forces the Lévy measure μ to satisfy

$$\int_{(0,\infty)} \bar{\mu}(x)\mathrm{d}x = \int_{(0,\infty)} x\mu(\mathrm{d}x) = \infty. \tag{4.10}$$

The condition (**A2**) is necessary to deal with the functional $v_t(s, \lambda, \xi)$ and obtain an upper bound for the speed of non-explosion when the sample paths of the Lévy process have a high running supremum (see Proposition 4.4.4 below for details). Furthermore, under condition (**A2**) and appealing to Proposition 4.1.2, it follows that there is a positive probability for the process with branching mechanism $\psi(\lambda) = -\phi(\lambda)$ to explode. Indeed, from (4.9) we observe

$$\int_{0+} \frac{1}{|-\phi(z)|}\mathrm{d}z \leqslant \int_{0+} \frac{1}{|C|z^{1+\beta}}\mathrm{d}z < \infty.$$

Recall that $\mathbb{P}_{(z,x)}$ (resp. the expectation operator $\mathbb{E}_{(z,x)}$) denotes the law of the couple (Z, ξ) starting from (z, x) where $z > 0$ and $x \in \mathbb{R}$. Roughly speaking, our aim is to show, under the above conditions, that $\kappa(1/t, 0)^{-1}\mathbb{P}_z(Z_t < \infty)$ has a limit as $t \uparrow \infty$. The result is formulated as follows.

Theorema 4.1.3 (Critical-explosion regime)**.** *Suppose that conditions* (**H1**)*,* (**A1**)*, and* (**A2**) *are satisfied. Then, for any* $z > 0$*, there exists* $0 < \mathfrak{C}_1(z) < \infty$ *such that*

$$\lim_{t\to\infty} \frac{1}{\kappa(1/t, 0)}\mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_1(z).$$

Recall that, $\kappa(\cdot, 0)$ corresponds to the Laplace exponent of the inverse local time L^{-1} associated to ξ. Then the result gives evidence that the asymptotic behaviour of the non-explosion probability is deeply related to the fluctuations of the Lévy environment ξ. Further, taking into account (4.7), we observe from Theorem 4.1.3 that, for any $z > 0$, there exists $0 < \mathfrak{C}_1(z) < \infty$ such that

$$\lim_{t\to\infty} \frac{t^{\rho}}{\ell(t)}\mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_1(z),$$

where ℓ is a slowly varying function at ∞ defined as

$$\ell(t) = \ell_1(1/t), \tag{4.11}$$

and we recall that ℓ_1 is the slowly varying function at 0^+ given in (4.7).

We now state our main last result in this section which is devoted to the speed of the non-explosion probability of CSBPs in a Lévy environment under the assumption that the environment drifts to $-\infty$. We recall that α_1 and π are the drift term and the Lévy measure of ξ, respectively. We introduce the following real function

$$A_\xi(x) := -\alpha_1 + \bar{\pi}^{(-)}(-1) + \int_{-x}^{-1} \bar{\pi}^{(-)}(y)\mathrm{d}y, \qquad \text{for} \quad x > 0,$$

where $\bar{\pi}^{(-)}(-x) = \pi(-\infty, -x)$. We also introduce the function

$$\widehat{\Phi}_\lambda(u) := \int_{(0,\infty)} \exp\{-\lambda e^u y\}\bar{\mu}(y)\mathrm{d}y, \tag{4.12}$$

Further, let us denote by $\mathtt{E}_1$ the exponential integral, i.e.,

$$\mathtt{E}_1(w) = \int_1^\infty \frac{e^{-wy}}{y}\mathrm{d}y, \qquad w > 0. \tag{4.13}$$

We can then formulate the following theorem.

Theorema 4.1.4 (Subcritical-explosion regime)**.** *Suppose that* $\Phi'_\xi(0+) < 0$ *and*

$$\int_{(a,\infty)} \frac{y}{A_\xi(y)}|\mathrm{d}\widehat{\Phi}_\lambda(y)| < \infty, \qquad \textit{for some} \quad a > 0. \tag{4.14}$$

Then, for any $z > 0$*, there exists* $0 < \mathfrak{C}_2(z) < \infty$ *such that*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_2(z).$$

In particular, if $\mathbb{E}^{(e)}\left[\widehat{\xi}_1\right] < \infty$*, then the integral condition* (4.14) *is equivalent to*

$$\int_0^\infty \mathtt{E}_1(\lambda y)\bar{\mu}(y)\mathrm{d}y < \infty. \tag{4.15}$$

Note that, if $\lambda y > 0$, we know that the following inequality for the exponential integral holds

$$\mathtt{E}_1(\lambda y) \leqslant e^{-\lambda y} \log\left(1 + \frac{1}{\lambda y}\right).$$

Therefore, in the case $\mathbb{E}^{(e)}\Big[\widehat{\xi}_1\Big] < \infty$, a simpler condition than (4.15) is the following

$$\int_0^\infty e^{-\lambda y} \log\left(1 + \frac{1}{\lambda y}\right) \bar{\mu}(y)\mathrm{d}y < \infty.$$

4.2 Conservativeness

This section is devoted to the proof of Theorem 4.1.1 by using the following extension of the classical Carathéodory's theorem for ordinary differential equations, that we state here for completeness.

Theorema 4.2.1 (Extended Carathéodory's existence theorem)**.** *Let $I = [-b, b]$ with $b > 0$. Assume that the function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

1. *the mapping $s \mapsto f(s, \theta)$ is measurable for each fixed $\theta \in \mathbb{R}$,*
2. *the mapping $\theta \mapsto f(s, \theta)$ is continuous for each fixed $s \in I$,*
3. *there exists a Lebesgue-integrable function m on the interval I such that*

$$|f(s, \theta)| \leqslant m(s)\Big(1 + |\theta|\Big), \quad (s, \theta) \in I \times \mathbb{R}.$$

Then there exists an absolutely continuous function $u(x)$ such that

$$u(x) = \int_0^x f(y, u(y))\mathrm{d}y, \qquad x \in I. \tag{4.16}$$

For a proof of this result the reader is referred to Person [65, Theorems 1.1, 2.1 and 2.3].

Proof of Theorem 4.1.1. The first part of the proof follows from similar arguments as those used in [6] and [58] in the case of finite mean (i.e., when $|\psi(0+)| < \infty$) whenever we have found an a.s. solution of the backward differential equation (4.5). In order to do so, we will appeal to the extended version of the Carathéodory's existence Theorem 4.2.1.

Fix $\omega \in \Omega^{(e)}$ and $t, \lambda \geqslant 0$. Denote by $f_\phi : [0, t] \times \mathbb{R} \to [-\infty, 0]$ the following function

$$f_\phi(s, \theta) = e^{\xi_s(\omega)}\psi\Big(\theta e^{-\xi_s(\omega)}\Big) = -e^{\xi_s(\omega)}\phi\Big(\theta e^{-\xi_s(\omega)}\Big).$$

In the following we omit the notation ω for the sake of brevity. First, we observe that the mapping $s \mapsto f_\phi(s, \theta)$ is measurable for each fixed $\theta \in \mathbb{R}$. Indeed, the process $\xi = (\xi_s,\ s \geqslant 0)$ is $(\mathcal{F}_t^{(e)})_{t\geqslant 0}$-adapted with càdlàg paths, this implies that it is progressively measurable. More precisely, the application $(s,\omega) \mapsto \xi_s$ is $\mathcal{B}([0,t])\otimes\mathcal{F}_t^{(e)}$-measurable. It follows that the mapping $s \mapsto f_\phi(s,\theta)$ is $\mathcal{B}([0,t])$-measurable for each fixed $\theta \in \mathbb{R}$ and $\omega \in \Omega^{(e)}$. Furthermore, we have that the function $\theta \mapsto f_\phi(s,\theta)$ is continuous for each fixed $s \in [0,t]$ due to the continuity of the function ψ. Therefore, according to Theorem 4.2.1, the proof is complete once we show that there exists an integrable function m_ϕ on $[0,t]$, such that, for any $(s,\theta) \in [0,t]\times\mathbb{R}$

$$|f_\phi(s,\theta)| \leqslant m_\phi(s)\Big(1+|\theta|\Big).$$

Note that, for any $(s,\theta) \in [0,t]\times\mathbb{R}$, we have

$$\begin{aligned}|f_\phi(s,\theta)| &= \Big|e^{\xi_s}\phi\Big(\theta e^{-\xi_s}\Big)\Big| = e^{\xi_s}\phi\Big(\theta e^{-\xi_s}\Big)\mathbf{1}_{\{\xi_s\geqslant 0\}} + e^{\xi_s}\phi\Big(\theta e^{-\xi_s}\Big)\mathbf{1}_{\{\xi_s<0\}}\\ &\leqslant e^{\xi_s}\phi(\theta)\mathbf{1}_{\{\xi_s\geqslant 0\}} + e^{\xi_s}\phi\Big(\theta e^{-\xi_s}\Big)\mathbf{1}_{\{\xi_s<0\}},\end{aligned}$$

where in the last inequality, we have used that ϕ is an increasing function. Now, since ϕ is a concave function it is well-known that for any $\theta > 0$ and $k > 1$, we have $\phi(\theta) \leqslant k\phi(\theta/k)$ (see for instance the proof of [10, Proposition III. 1]). In particular, this inequality implies,

$$|f_\phi(s,\theta)| \leqslant e^{\xi_s}\phi(\theta)\mathbf{1}_{\{\xi_s\geqslant 0\}} + \phi(\theta)\mathbf{1}_{\{\xi_s<0\}} \leqslant \max\{e^{\xi_s}\mathbf{1}_{\{\xi_s\geqslant 0\}}, \mathbf{1}_{\{\xi_s<0\}}\}\phi(\theta).$$

On the other hand, from (4.2) we have that $\phi(\theta) = \theta g(\theta)$, where g is the decreasing function

$$g(\theta) = \int_{(0,\infty)} e^{-\theta x}\bar{\mu}(x)\mathrm{d}x, \tag{4.17}$$

where we recall, from the observation above Theorem 4.1.1, that $\delta = 0$. In addition, observe that

$$g(0) = \int_{(0,\infty)} \bar{\mu}(x)\mathrm{d}x = \int_{(0,\infty)} x\mu(\mathrm{d}x) = \infty.$$

Thus, for any $d \in (0, g(0))$ there exists $\theta^* > 0$ such that $g(\theta^*) - d = 0$. In other words, θ^* is the largest root of $\phi(\theta) - \theta d$. It follows that, there exists $d \in (0, g(0))$ such that $\phi(\theta) \leqslant d + \theta$ for any $\theta > 0$. It turns out that,

$$|f(s,\theta)| \leqslant m_\phi(s)\Big(1+|\theta|\Big), \qquad \text{for all} \qquad (s,\theta) \in [0,t]\times\mathbb{R},$$

where

$$m_\phi(s) := (1\vee d)\max\{e^{\xi_s}\mathbf{1}_{\{\xi_s\geqslant 0\}}, \mathbf{1}_{\{\xi_s<0\}}\}, \qquad s \in [0,t].$$

Note that m_ϕ is an integrable function on $[0,t]$ since the Lévy process ξ has càdlàg paths. Finally, thanks to Theorem 4.2.1, there exists an a.s. solution of (4.5).

□

The functional $v_t(s,\lambda,\xi)$ has some useful monotonicity properties as is stated in the following lemma. In the forthcoming sections, we will make use of these properties.

Lemma 4.2.2. *For any $\lambda \geqslant 0$ and $t \geqslant 0$, the mapping $s \mapsto v_t(s,\lambda,\xi)$ is decreasing on $[0,t]$. For any $s \in [0,t]$, the mapping $\lambda \mapsto v_t(s,\lambda,\xi)$ is increasing on $[0,\infty)$.*

Proof. Recall that we are assuming $\psi(\theta) = -\phi(\theta) \leqslant 0$. Then, from the backward differential equation (4.5), we see that the function $s \mapsto v_t(s,\lambda,\xi)$ is decreasing on $[0,t]$. Further, from (4.4) we observe that the mapping $\lambda \mapsto v_t(s,\lambda,\xi)$ is increasing on $[0,\infty)$. □

We conclude this section with the proof of Proposition 4.1.2. Before we do so, let us make an important remark about the non-explosion probability. More precisely, it is easy to deduce, by letting $\lambda \downarrow 0$ in (4.4) and with the help of the Monotone Convergence Theorem, that the non-explosion probability is given by

$$\mathbb{P}_{(z,x)}\Big(Z_t < \infty \mid \xi\Big) = \exp\left\{-z \lim_{\lambda\downarrow 0} v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0)\right\}, \quad z,t>0, \quad x \in \mathbb{R}. \tag{4.18}$$

With this in hand, we may now observe that the process Z is *conservative* if and only if

$$\lim_{\lambda\downarrow 0} v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0) = 0, \qquad \text{for all} \qquad t > 0. \tag{4.19}$$

Proof of Proposition 4.1.2. Fix $t > 0$ and recall that $(\underline{\xi}_t, t \geqslant 0)$ and $(\overline{\xi}_t, t \geqslant 0)$ denote the running infimum and supremum of the process ξ, respectively. First, we assume that the branching mechanism ψ satisfies (4.6), that is to say,

$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \int_{0+} \frac{1}{|-\phi(z)|}\mathrm{d}z = \int_{0+} \frac{1}{\phi(z)}\mathrm{d}z = \infty.$$

From Theorem 4.1.1, we see that the backward differential equation (4.5) can be rewritten as follows

$$t = \int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\xi_s}\psi\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)} = -\int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\xi_s}\phi\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)}.$$

Now, we recall that ϕ is an increasing and non-negative function. Then appealing to the definition of the running infimum and supremum of ξ, we observe that the following inequality holds

$$-t = \int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\xi_s}\phi\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)} \leqslant \int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\underline{\xi}_t}\phi\Big(e^{-\overline{\xi}_t}v_t(s,\lambda,\xi)\Big)}$$
$$= \frac{e^{\overline{\xi}_t}}{e^{\underline{\xi}_t}} \int_{e^{-\overline{\xi}_t}v_t(0,\lambda,\xi)}^{e^{-\overline{\xi}_t}\lambda} \frac{1}{\phi(z)}\mathrm{d}z,$$

where in the last equality we have used change of variables $z = e^{-\overline{\xi}_t}v_t(s,\lambda,\xi)$. Next, letting $\lambda \downarrow 0$ in the previous inequality, we get

$$\frac{e^{\overline{\xi}_t}}{e^{\underline{\xi}_t}} \int_0^{e^{-\overline{\xi}_t}\lim\limits_{\lambda\downarrow 0} v_t(0,\lambda,\xi)} \frac{1}{\phi(z)}\mathrm{d}z \leqslant t. \tag{4.20}$$

Thus, taking into account our assumption, we are forced to conclude that

$$\lim_{\lambda\downarrow 0} v_t(0,\lambda,\xi) = 0. \tag{4.21}$$

In other words, the process is conservative.

On the other hand, we assume that the process is conservative or equivalently that (4.21) holds. We will proceed by contradiction, we suppose that

$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \int_{0+} \frac{1}{\phi(z)}\mathrm{d}z < \infty.$$

Similar to the above arguments, we deduce that

$$-t = \int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\xi_s}\phi\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)} \geqslant \int_0^t \frac{\mathrm{d}v_t(s,\lambda,\xi)}{e^{\overline{\xi}_t}\phi\Big(e^{-\underline{\xi}_t}v_t(s,\lambda,\xi)\Big)}$$
$$= \frac{e^{\underline{\xi}_t}}{e^{\overline{\xi}_t}} \int_{e^{-\underline{\xi}_t}v_t(0,\lambda,\xi)}^{e^{-\underline{\xi}_t}\lambda} \frac{1}{\phi(z)}\mathrm{d}z.$$

Taking $\delta > 0$ sufficiently small, we see

$$t \leqslant \frac{e^{\underline{\xi}_t}}{e^{\overline{\xi}_t}} \int_{e^{-\underline{\xi}_t}\lambda}^{\delta} \frac{1}{\phi(z)}\mathrm{d}z - \frac{e^{\underline{\xi}_t}}{e^{\overline{\xi}_t}} \int_{e^{-\underline{\xi}_t}v_t(0,\lambda,\xi)}^{\delta} \frac{1}{\phi(z)}\mathrm{d}z.$$

Hence, taking $\lambda \downarrow 0$ in the above inequality, we have $t \leqslant 0$, which is a contradiction. Therefore, we deduce that the branching mechanism ψ satisfies (4.6). $\square$

4.3 CSBPs in a conditioned Lévy environment

As mentioned earlier, the asymptotic behaviour of the non-explosion probability is related to the running supremum of the environment ξ. In order to study this relationship we recall from Section 3.5 that $\mathbb{P}^{\downarrow}_{(z,x)}$ corresponds to the law of (Z,ξ) conditioned to be negative.

The following convergence result is crucial for Theorem 4.1.3. Similar ideas to those used in the proof can be found in [3, Lemma 2.5] and also in [7, Lemma 1] for the discrete and continuous setting, respectively. We provide its proof for the sake of completeness.

Lemma 4.3.1. *Fix $z>0$, $x<0$ and assume that Spitzer's condition* (**H1**) *holds. Let R_s be a bounded real-valued $\mathcal{F}_s$-measurable random variable. Then*

$$\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_s \mid \overline{\xi}_t<0\Big]=\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_s\Big].$$

More generally, let $(R_t, t\geqslant 0)$ be a uniformly bounded real-valued process adapted to the filtration $(\mathcal{F}_t, t\geqslant 0)$, which converges $\mathbb{P}^{\downarrow}_{(z,x)}$-a.s. to some random variable R_∞. Then

$$\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_t \mid \overline{\xi}_t<0\Big]=\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big].$$

Proof. Fix $z>0$ and $x<0$. Let us proof the first claim. For $s,h\geqslant 0$, conditioning on $\mathcal{F}_s$ and appealing to the Markov property, we have,

$$\mathbb{E}_{(z,x)}\Big[R_s \mid \overline{\xi}_{s+h}<0\Big]=\mathbb{E}_{(z,x)}\left[R_s\frac{\mathbb{P}^{(e)}_{\xi_s}\Big(\overline{\xi}_h<0\Big)}{\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_{s+h}<0\Big)}\mathbf{1}_{\{\overline{\xi}_s<0\}}\right]. \tag{4.22}$$

As we had mentioned in Section 4.1, under Spitzer's condition (**H1**) we have, from Theorem 14 in [10], that the Laplace exponent $\kappa(\cdot,0)$ is regularly varying at 0^+ with index ρ. Then this implies that there exists a slowly varying function ℓ at ∞ such that

$$\kappa(q,0)=\ell(1/q)q^{\rho}.$$

Now, with this observation in mind, it is worth recalling here the asymptotic behaviour of the probability that the Lévy process ξ remains negative satisfies

$$\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_t<0\Big)\sim\pi^{-1/2}U(-x)\ell(t)t^{-\rho},\qquad \text{as}\quad t\to\infty.$$

Then, for $\epsilon>0$ there exists a constant $N_1>0$ (which depends on ϵ) such that the following inequality holds for all $h \geqslant N_1$,

$$\frac{\mathbb{P}^{(e)}_{\xi_s}\left(\overline{\xi}_h < 0\right)}{\mathbb{P}^{(e)}_{x}\left(\overline{\xi}_{s+h} < 0\right)} \leqslant \frac{(1+\epsilon)}{(1-\epsilon)}\frac{\ell(h)}{\ell(s+h)}\left(\frac{h}{s+h}\right)^{-\rho}\frac{U(-\xi_s)}{U(-x)}.$$

Moreover, by Potter's Theorem (see Theorem 1.5.6 in Bingham [15]), for any $A > 1$ and $\delta > 0$ there exists a constant $N_2 > 0$ such that

$$\frac{\ell\,(h)}{\ell\,(s+h)} \leqslant A\max\left\{\left(\frac{h}{s+h}\right)^{\delta}, \left(\frac{h}{s+h}\right)^{-\delta}\right\}, \qquad \text{for} \quad h \geqslant N_2.$$

Therefore, we deduce that for $h \geqslant N = \max\{N_1, N_2\}$,

$$\frac{\mathbb{P}^{(e)}_{\xi_s}\left(\overline{\xi}_h < 0\right)}{\mathbb{P}^{(e)}_{x}\left(\overline{\xi}_{s+h} < 0\right)} \leqslant \frac{(1+\epsilon)}{(1-\epsilon)}\left(1+\frac{s}{N}\right)^{\delta+\rho}\frac{U(-\xi_s)}{U(-x)}. \tag{4.23}$$

Since the renewal function U is finite and R_s is a bounded random variable, then we can apply the Dominated Convergence Theorem in (4.22). Hence, the first claim now follows as a consequence of Theorem VI.18 in [10]. Indeed, we have

$$\lim_{h\to\infty}\frac{\mathbb{P}^{(e)}_{\xi_s}\left(\overline{\xi}_h < 0\right)}{\mathbb{P}^{(e)}_{x}\left(\overline{\xi}_{s+h} < 0\right)} = \frac{U(-\xi_s)}{U(-x)}.$$

For the second claim let $\gamma > 1$. Using again the Markov property at time t, we obtain for $s \leqslant t$,

$$\left|\mathbb{E}_{(z,x)}\left[R_t - R_s \mid \overline{\xi}_{\gamma t} < 0\right]\right| \leqslant \mathbb{E}_{(z,x)}\left[|R_t - R_s|\frac{\mathbb{P}^{(e)}_{\xi_t}\left(\overline{\xi}_{(\gamma-1)t} < 0\right)}{\mathbb{P}^{(e)}_{x}\left(\overline{\xi}_{\gamma t} < 0\right)}\mathbf{1}_{\{\overline{\xi}_t < 0\}}\right].$$

Next, once again appealing to Potter's Theorem, we obtain, for any $A_2 > 1$ and $\delta_2 > 0$, that there exists a constant $N_3 > 0$ such that

$$\frac{\ell\,(t(\gamma-1))}{\ell\,(t\gamma)} \leqslant C(A_2, \delta_2, \gamma), \qquad \text{for all} \quad t \geqslant N_3,$$

where

$$C(A_2, \delta_2, \gamma) := A_2\max\left\{\left(\frac{\gamma-1}{\gamma}\right)^{\delta_2}, \left(\frac{\gamma-1}{\gamma}\right)^{-\delta_2}\right\}.$$

Similarly as in (4.23), we get

$$\frac{\mathbb{P}^{(e)}_{\xi_t}\Big(\overline{\xi}_{(\gamma-1)t}<0\Big)}{\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_{\gamma t}<0\Big)}\leqslant\frac{(1+\epsilon)}{(1-\epsilon)}C(A_2,\delta_2,\gamma)\left(\frac{\gamma-1}{\gamma}\right)^{-\rho}\frac{U(-\xi_t)}{U(-x)}.$$

This in turn implies that

$$\begin{aligned}\Big|\mathbb{E}_{(z,x)}\Big[R_t-R_s\mid\overline{\xi}_{\gamma t}<0\Big]\Big|&\leqslant C_\gamma\mathbb{E}_{(z,x)}\left[|R_t-R_s|\frac{U(-\xi_t)}{U(-x)}\mathbf{1}_{\{\overline{\xi}_t<0\}}\right]\\&=C_\gamma\mathbb{E}^{\downarrow}_{(z,x)}\Big[|R_t-R_s|\Big],\end{aligned}$$

where in the last equality we have used the definition of the measure $\mathbb{P}^{\downarrow}_{(z,x)}$ given in (3.25) and

$$C_\gamma:=\frac{(1+\epsilon)}{(1-\epsilon)}C(A_2,\delta_2,\gamma)\left(\frac{\gamma-1}{\gamma}\right)^{-\rho}.$$

Letting first $t\to\infty$ and then $s\to\infty$ in the previous inequality, the right-hand side vanishes by the Dominated Convergence Theorem and since $(R_t,t\geqslant 0)$ is a uniformly bounded process which converges $\mathbb{P}^{\downarrow}_{(z,x)}$-a.s. to R_∞. Thus

$$\lim_{s\to\infty}\lim_{t\to\infty}\Big|\mathbb{E}_{(z,x)}\Big[R_t-R_s\mid\overline{\xi}_{\gamma t}<0\Big]\Big|=0,$$

which yields

$$\lim_{s\to\infty}\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_t-R_s\mid\overline{\xi}_{\gamma t}<0\Big]=0.$$

Now, appealing to the first part of this lemma, we get

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_t\mid\overline{\xi}_{\gamma t}<0\Big]&=\lim_{s\to\infty}\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_s\mid\overline{\xi}_{\gamma t}<0\Big]\\&=\lim_{s\to\infty}\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_s\Big]=\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big].\end{aligned}$$

Moreover,

$$\lim_{t\to\infty}\frac{\mathbb{E}_{(z,x)}\Big[R_t\ ,\ \overline{\xi}_{\gamma t}<0\Big]}{\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_t<0\Big)}=\lim_{t\to\infty}\mathbb{E}_{(z,x)}\Big[R_t\mid\overline{\xi}_{\gamma t}<0\Big]\frac{\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_{\gamma t}<0\Big)}{\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_t<0\Big)}=\gamma^{-\rho}\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big].$$

Since γ may be chosen arbitrarily close to 1, we have

$$\mathbb{E}_{(z,x)}\Big[R_t\ ,\ \overline{\xi}_{\gamma t}<0\Big]=\Big(\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big]+o(1)\Big)\mathbb{P}^{(e)}_{x}\Big(\overline{\xi}_t<0\Big).\qquad(4.24)$$

Once again, since $(R_t,t\geqslant 0)$ is a uniformly bounded process, there exists a positive constant C_2 such that $|R_t| \leqslant C_2$. In addition, observe that, given any $\epsilon > 0$ and t sufficiently large, we have

$$\begin{aligned}\Big|\mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_t<0\}}\Big] - \mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_{\gamma t}<0\}}\Big]\Big| &\leqslant C_2\Big(\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big) - \mathbb{P}^{(e)}_x\Big(\overline{\xi}_{\gamma t} < 0\Big)\Big)\\ &\leqslant C_2\left(1 - \frac{\mathbb{P}^{(e)}_x\Big(\overline{\xi}_{\gamma t} < 0\Big)}{\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big)}\right)\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big)\\ &\leqslant C_2\left(1 - \frac{1-\epsilon}{1+\epsilon}\gamma^{-\rho}\right)\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big).\end{aligned}$$

Hence, since ϵ is arbitrary close to 0 and γ may be chosen arbitrarily close to 1, we obtain

$$\mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_t<0\}}\Big] - \mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_{\gamma t}<0\}}\Big] = o(1)\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big). \tag{4.25}$$

Finally, putting everything together, this is, from (5.3) and (4.25), we deduce that

$$\begin{aligned}\mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_t<0\}}\Big] - \mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big]\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big) &= \mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_t<0\}}\Big] - \mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_{\gamma t}<0\}}\Big]\\ &\quad + \mathbb{E}_{(z,x)}\Big[R_t \mathbf{1}_{\{\overline{\xi}_{\gamma t}<0\}}\Big] - \mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big]\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big)\\ &= o\Big(\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big)\Big).\end{aligned}$$

The second claim is now complete. □

Recall from Theorem 4.1.1 that the quenched law of the process $(Z_t e^{-\xi_t}, t \geqslant 0)$ is completely characterised by the functional $v_t(s, \lambda, \xi)$. In the case of conditioned environment we have a similar result. We formalize this in the following lemma.

Lemma 4.3.2. *For each $z > 0$, $x < 0$ and $\lambda \geqslant 0$, we have*

$$\mathbb{E}^{\downarrow}_{(z,x)}\Big[\exp\Big\{-\lambda Z_t e^{-\xi_t}\Big\}\Big] = \mathbb{E}^{(e),\downarrow}_x\Big[\exp\{-z v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0)\}\Big]. \tag{4.26}$$

In particular,

$$\mathbb{P}^{\downarrow}_{(z,x)}(Z_t < \infty) = \mathbb{E}^{(e),\downarrow}_x\Big[\exp\{-z v_t(0, 0, \xi - \xi_0)\}\Big].$$

Essentially the proof mimics the steps of [7, Proposion 2]. However, we present it here for the sake of completeness.

Proof. Let $z > 0$ and $x < 0$. Using the law of a CSBP in a Lévy environment ξ conditioned to stay negative given in (3.25) followed by conditioning on the environment, we obtain for every $\lambda \geqslant 0$,

$$\begin{aligned}\mathbb{E}^{\downarrow}_{(z,x)}\Big[\exp\Big\{-\lambda Z_t e^{-\xi_t}\Big\}\Big] &= \frac{1}{U(-x)}\mathbb{E}_{(z,x)}\Big[U(-\xi_t)\mathbf{1}_{\{\bar{\xi}_t<0\}}e^{-\lambda Z_t e^{-\xi_t}}\Big]\\ &= \frac{1}{U(-x)}\mathbb{E}_{(z,0)}\Big[U(-\xi_t-x)\mathbf{1}_{\{\bar{\xi}_t<-x\}}\mathbb{E}_{(z,0)}\Big[e^{-\lambda e^{-x}Z_t e^{-\xi_t}}\Big|\xi\Big]\Big]\\ &= \frac{1}{U(-x)}\mathbb{E}_{(z,0)}\Big[U(-\xi_t-x)\mathbf{1}_{\{\bar{\xi}_t<-x\}}e^{-zv_t(0,\lambda e^{-x},\xi)}\Big]\\ &= \mathbb{E}^{(e),\downarrow}_{x}\Big[\exp\{-zv_t(0,\lambda e^{-\xi_0},\xi-\xi_0)\}\Big].\end{aligned}$$

Further, by letting $\lambda \downarrow 0$, we deduce that

$$\mathbb{P}^{\downarrow}_{(z,x)}(Z_t<\infty) = \mathbb{E}^{(e),\downarrow}_{x}\Big[\exp\{-zv_t(0,0,\xi-\xi_0)\}\Big].$$

$\square$

The following lemma states that, with respect to $\mathbb{P}^{\downarrow}_{(z,x)}$, the population has positive probability to be finite forever. In other words, Z has a positive probability to be finite when the running supremum of the Lévy environment is negative. The statement holds under a moment condition on the descending ladder height process associated to ξ. Further, note that such behaviour is similar to the behaviour in the subcritical-explosion regime (i.e., when the environment drifts to $-\infty$) given by Palau et al. [59] for a CSBP in Lévy environment with a stable branching mechanism.

Lemma 4.3.3. *Assume that the branching mechanism satisfies condition* (**A2**)*. Also suppose that the Lévy process ξ satisfies condition* (**A1**)*. Then, for $z > 0$ and $x < 0$, we have*

$$\lim_{t\to\infty}\mathbb{P}^{\downarrow}_{(z,x)}(Z_t<\infty) > 0.$$

Before we proceed with the proof, let us recall from the observation above Theorem 4.1.1 that we may take $\delta = 0$.

Proof. Let $z > 0$ and $x < 0$. From Lemma (4.3.2), we already know the formula,

$$\lim_{t\to\infty}\mathbb{P}^{\downarrow}_{(z,x)}(Z_t<\infty) = \lim_{t\to\infty}\mathbb{E}^{(e),\downarrow}_{x}\Big[\exp\{-zv_t(0,0,\xi-\xi_0)\}\Big].$$

Then, in order to deduce our result we will to show that

$$\lim_{t\to\infty} v_t(0,0,\xi-\xi_0) < \infty, \quad \mathbb{P}^{(e),\downarrow}_{x} - \text{a.s.} \tag{4.27}$$

Now, from Lemma 4.2.2, we see that the mapping $s \mapsto v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0)$ is decreasing and thus $v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) \geqslant \lambda e^{-\xi_0}$ for all $s \in [0,t]$. It follows that, for $s \in [0,t]$ and $\lambda \geqslant 0$,

$$\begin{aligned}\frac{\partial}{\partial s} v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) &= e^{\xi_s - \xi_0} \psi\Big(v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) e^{-\xi_s + \xi_0}\Big) \\ &= -v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) \int_{(0,\infty)} \exp\{-v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) e^{-\xi_s + \xi_0} z\} \bar{\mu}(z) \mathrm{d}z \\ &\geqslant -v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) \int_{(0,\infty)} \exp\{-\lambda e^{-\xi_0} e^{-\xi_s + \xi_0} z\} \bar{\mu}(z) \mathrm{d}z,\end{aligned}$$

Therefore by integrating,

$$\log v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0) \leqslant \log(\lambda e^{-\xi_0}) + \int_0^t \int_{(0,\infty)} \exp\{-\lambda e^{-\xi_s} z\} \bar{\mu}(z) \mathrm{d}z \mathrm{d}s.$$

Moreover, from Lemma 4.2.2, we have that the mapping $\lambda e^{-\xi_0} \mapsto v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0)$ is increasing. It tuns out that,

$$v_t(0, 0, \xi - \xi_0) \leqslant v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0) \leqslant \lambda e^{-\xi_0} \exp\left(\int_0^t \int_{(0,\infty)} \exp\{-\lambda e^{-\xi_s} z\} \bar{\mu}(z) \mathrm{d}z \mathrm{d}s\right).$$

Let us denote,

$$\Phi_\lambda(u) := \int_{(0,\infty)} \exp\{-\lambda e^{-u} z\} \bar{\mu}(z) \mathrm{d}z.$$

It then follows,

$$\lim_{t \to \infty} v_t(0, 0, \xi - \xi_0) \leqslant \lambda e^{-\xi_0} \exp\left(\int_0^\infty \Phi_\lambda(\xi_s) \mathrm{d}s\right).$$

Now, if the right-hand side above is finite $\mathbb{P}_x^{(e),\downarrow}$ – a.s. then (4.27) holds. The result is thus proved once we show that

$$\mathbb{E}_x^{(e),\downarrow}\left[\int_0^\infty \Phi_\lambda(\xi_s) \mathrm{d}s\right] < \infty.$$

First, with the help of Fubini's Theorem and the definition of the measure $\mathbb{P}_x^{(e),\downarrow}$, we obtain

$$\begin{aligned}\mathbb{E}_x^{(e),\downarrow}\left[\int_0^\infty \Phi_\lambda(\xi_s) \mathrm{d}s\right] &= \frac{1}{U(-x)} \int_0^\infty \widehat{\mathbb{E}}_{-x}^{(e)}\Big[U(\xi_s) \Phi_\lambda(\xi_s) \mathbf{1}_{\{\underline{\xi}_s > 0\}}\Big] \mathrm{d}s \\ &= \frac{1}{U(-x)} \widehat{\mathbb{E}}_{-x}^{(e)}\left[\int_0^{\tau_0^-} U(\xi_s) \Phi_\lambda(\xi_s) \mathrm{d}s\right].\end{aligned}$$

Now, applying Theorem VI.20 in Bertoin [10] to the dual process $\widehat{\xi} = -\xi$ and the function $f(y) = U(y)\Phi_\lambda(y),\ y \geqslant 0$, we deduce that, there exists a constant $k > 0$ such that

$$\widehat{\mathbb{E}}^{(e)}_{-x}\left[\int_0^{\tau_0^-} U(\xi_s)\Phi_\lambda(\xi_s)\mathrm{d}s\right] = k\int_{[0,\infty)} \mathrm{d}\widehat{U}(y)\int_{[0,-x]} \mathrm{d}U(z)U(y-x-z)\Phi_\lambda(y-x-z).$$

For the sake of simplicity we take $k = 1$ (we may choice a normalisation of local time in order to have $k = 1$). Observe that, for any $z \in [0, -x]$ and $y \geqslant 0$, we have $y - x - z \leqslant y - x$. Further, since $U(\cdot)$ and $\Phi_\lambda(\cdot)$ are increasing functions, we deduce that $U(y-x-z) \leqslant U(y-x)$ and $\Phi_\lambda(y-x-z) \leqslant \Phi_\lambda(y-x)$, which implies

$$\begin{aligned}\widehat{\mathbb{E}}^{(e)}_{-x}\left[\int_0^{\tau_0^-} U(\xi_s)\Phi_\lambda(\xi_s)\mathrm{d}s\right] &\leqslant \int_{[0,\infty)} \mathrm{d}\widehat{U}(y)\int_{[0,-x]} \mathrm{d}U(z)U(y-x)\Phi_\lambda(y-x)\\ &= (U(-x)-U(0))\int_{[0,\infty)} \mathrm{d}\widehat{U}(y)U(y-x)\Phi_\lambda(y-x).\end{aligned}$$

On the other hand, note that the function Φ_λ can be also rewritten as follows,

$$\begin{aligned}\Phi_\lambda(u) &= \int_{(0,\infty)} \exp\{-\lambda e^{-u}z\}\bar{\mu}(z)\mathrm{d}z = \int_{[0,\infty)} \mu(\mathrm{d}y)\int_0^y \exp\{-\lambda e^{-u}z\}\mathrm{d}z\\ &= \int_{[0,\infty)} \frac{1-\exp\{-\lambda e^{-u}y\}}{\lambda e^{-u}}\mu(\mathrm{d}y).\end{aligned}$$

Hence putting all pieces together, we obtain

$$\begin{aligned}\mathbb{E}^{(e),\downarrow}_x\left[\int_0^{\infty} \Phi_\lambda(\xi_s)\mathrm{d}s\right] &= \frac{1}{U(-x)}\widehat{\mathbb{E}}^{(e)}_{-x}\left[\int_0^{\tau_0^-} U(\xi_s)\Phi_\lambda(\xi_s)\mathrm{d}s\right]\\ &\leqslant \int_{(0,\infty)} \Delta(z)\mu(\mathrm{d}z),\end{aligned}$$

where

$$\Delta(z) := \int_{[0,\infty)} \mathrm{d}\widehat{U}(y)U(y-x)\left[\frac{1-\exp\{-\lambda e^{x-y}z\}}{\lambda e^{x-y}}\right]. \tag{4.28}$$

We claim that the integral of $z \mapsto \Delta(z)\mu(\mathrm{d}z)$ is finite under condition (**A1**). In order to see this, we first note that, for $z \in (0, 1)$ we get

$$\Delta(z) \leqslant \int_{[0,\infty)} \mathrm{d}\widehat{U}(y)U(y-x)z \leqslant C_1 z\int_{[0,\infty)} \mathrm{d}\widehat{U}(y)y = C_1 z\widehat{\mathbb{E}}^{(e)}\Big[H_1\Big],$$

where C_1 is a finite constant that only depends on x. Thus, for $z \in (0, 1)$

$$\int_{(0,\infty)} \Delta(z)\mu(\mathrm{d}z) \leqslant C_1\widehat{\mathbb{E}}^{(e)}\Big[H_1\Big]\int_{(0,1)} z\mu(\mathrm{d}z).$$

Note that, the term in the right-hand side is finite due to condition (**A1**) and since μ is such that

$$\int_{(0,\infty)} (1 \wedge x)\mu(\mathrm{d}x) < \infty \tag{4.29}$$

Finally, for $z \in (1, \infty)$,

$$\Delta(z) \leqslant C_1 \int_{[0,\infty)} \mathrm{d}\widehat{U}(y) y \frac{1}{\lambda e^{x-y}} = C_1 \lambda^{-1} e^{-x} \int_{[0,\infty)} \mathrm{d}\widehat{U}(y) y e^{y} = \widehat{C}_1 \widehat{\mathbb{E}}^{(e)} \Big[H_1 e^{H_1} \Big],$$

where $\widehat{C}_1$ is a finite constant. This implies that

$$\int_{(0,\infty)} \Delta(z) \mu(\mathrm{d}z) \leqslant \widehat{C}_1 \widehat{\mathbb{E}}^{(e)} \Big[H_1 e^{H_1} \Big] \int_{(1,\infty)} \mu(\mathrm{d}z).$$

Once again, appealing to the condition (**A1**) and (4.29), we have that the above integral is also finite. This concludes the proof.

□

4.4 Critical-explosion regime

This section is devoted to the proof of Theorem 4.1.3. The general approach of our proof is to replace the event $\{Z_t < \infty\}$ by others events depending on the behaviour of the running supremum of the environment, which are easier to handle. This strategy has been used before to deal with the absorption event in the discrete setting as well as in the continuous setting (see for instance [3] and [7]), where the survival event is splitted when the running infimum of the environment is either positive or negative. Since we are studying the non-explosion event, we split it when the running supremum is either positive or negative.

Before we prove Theorem 4.1.3, we introduce several results that are required for our arguments. Lemmas 4.3.1 and 4.3.3 allows us to establish the first result which describes the limit of the non-explosion probability when the associated environment is conditioned to be negative.

Proposition 4.4.1. *Suppose that conditions* (**H1**), (**A1**) *and* (**A2**) *are satisfied. Then for every* $z > 0$ *and* $x < 0$, *there exists* $0 < c(z, x) < \infty$ *such that*

$$\lim_{t\to\infty} \frac{1}{\kappa(1/t, 0)} \mathbb{P}_{(z,x)} \Big(Z_t < \infty,\ \overline{\xi}_t < 0 \Big) = c(z, x) U(-x). \tag{4.30}$$

Proof. We begin by defining the decreasing sequence of events $A_t = \{Z_t < \infty\}$ for $t \geqslant 0$, and also the event $A_\infty = \{\forall t \geqslant 0,\ Z_t < \infty\}$. Now, observe that

$$\lim_{t\to\infty} A_t = A_\infty.$$

Let $(R_t := \mathbf{1}_{A_t},\ t \geqslant 0)$ be a uniformly bounded process adapted to the filtration $(\mathcal{F}_t, t \geqslant 0)$. Note that, the process $(R_t, t \geqslant 0)$ converges $\mathbb{P}^{\downarrow}_{(z,x)}$-a.s. to a random variable $R_\infty = \mathbf{1}_{A_\infty}$. Then, by appealing to Lemma 4.3.1, we have

$$\lim_{t\to\infty} \mathbb{E}_{(z,x)}\Big[R_t \mid \overline{\xi}_t < 0\Big] = \mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big]. \tag{4.31}$$

Therefore, by using the asymptotic behaviour of the probability that the Lévy process ξ remains negative given in (4.8), we get

$$\begin{aligned} \mathbb{P}_{(z,x)}\Big(Z_t < \infty,\ \overline{\xi}_t < 0\Big) &= \mathbb{E}_{(z,x)}\Big[R_t \mid \overline{\xi}_t < 0\Big]\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big) \\ &\sim c(z,x)U(-x)\kappa(1/t,0), \quad \text{as} \quad t\to\infty, \end{aligned}$$

where $c(z,x) := \mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big]/\sqrt{\pi}$. Furthermore, from Lemma 4.3.3, we have

$$\mathbb{E}^{\downarrow}_{(z,x)}\Big[R_\infty\Big] = \mathbb{P}^{\downarrow}_{(z,x)}\Big(\forall t \geqslant 0,\ Z_t < \infty\Big) = \lim_{t\to\infty} \mathbb{P}^{\downarrow}_{(z,x)}(Z_t < \infty) > 0,$$

which completes the proof. □

Lemma 4.4.2. *Let $x < 0$ and assume that condition* (**H1**) *holds. Thus for any $s \leqslant t$, as t and s goes to ∞, we have,*

$$\widehat{\mathbb{P}}^{(e)}\Big(s < \tau_x^- \leqslant t\Big) \leqslant C_3\left(C_4\left(\frac{t}{s}\right)^{\eta+\rho} - 1\right)U(-x)t^{-\rho}\ell(t),$$

where $C_3, C_4 > 0$, $\eta > 0$ and ℓ is the slowly varying function at ∞ given in (4.11).

Proof. Let $x < 0$ and $s \leqslant t$. Let us begin by noting that,

$$\begin{aligned} \widehat{\mathbb{P}}^{(e)}\Big(s < \tau_x^- \leqslant t\Big) &= \widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > s\Big) - \widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big) \\ &= \widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big)\left(\frac{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > s\Big)}{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big)} - 1\right). \end{aligned} \tag{4.32}$$

Now, recall that under Spitzer's condition (**H1**), the function $\kappa(\cdot, 0)$ is regularly varying at 0. To be more precise, from (4.8), we get

$$\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big) \sim \frac{U(-x)}{\sqrt{\pi}}t^{-\rho}\ell(t), \quad \text{as} \quad t\to\infty,$$

where ℓ is the slowly varying function at ∞ given in (4.11). Hence, for t and s large enough, we have

$$\frac{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > s\Big)}{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big)} \leqslant C_1\left(\frac{s}{t}\right)^{-\rho}\frac{\ell(s)}{\ell(t)},$$

where C_1 is a positive constant. On the other hand, according to Potter's Theorem in

[15] we deduce that, for any $A > 1$ and $\eta > 0$ there exists $t_1 = t_1(A, \eta)$ such that

$$\frac{\ell(s)}{\ell(t)} \leqslant A \max\left\{\left(\frac{s}{t}\right)^{\eta}, \left(\frac{s}{t}\right)^{-\eta}\right\}, \qquad t \geqslant s \geqslant t_1.$$

Therefore, for $t \geqslant s \geqslant t_1$

$$\frac{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > s\Big)}{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big)} \leqslant C_2 \left(\frac{s}{t}\right)^{-\rho} \left(\frac{s}{t}\right)^{-\eta} \leqslant C_2 \left(\frac{t}{s}\right)^{\eta+\rho},$$

where the constants $C_2 > 0$ is adjusted properly. Now plugging the later inequality back into (4.32), we get, as it was claimed,

$$\begin{aligned}\widehat{\mathbb{P}}^{(e)}\Big(s < \tau_x^- \leqslant t\Big) &\leqslant \widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big) \left(\frac{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > s\Big)}{\widehat{\mathbb{P}}^{(e)}\Big(\tau_x^- > t\Big)} - 1\right) \\ &\leqslant C_3 \left(C_4 \left(\frac{t}{s}\right)^{\eta+\rho} - 1\right) U(-x) t^{-\rho} \ell(t), \qquad \text{as} \quad s, t \to \infty,\end{aligned}$$

where C_3 and C_4 are strictly positive constants. □

Recall that $\mathtt{I}_{0,t}(\beta\xi)$ is the exponential functional of the Lévy process $\beta\xi$ defined in (3.29). According to Theorem 1 in Bertoin and Yor [12], the exponential functional $\mathtt{I}_{0,\infty}(\beta\xi)$ is a.s. finite if and only if ξ drifs to $-\infty$. The following result will be useful to control the probability of non-explosion under the event that $\{\overline{\xi}_t > 0\}$.

Lemma 4.4.3. *Let $\beta \in (-1, 0)$, $C < 0$ and $y > 0$. Assume that condition* (**H1**) *holds. Then, there exists a constant $C_\beta(y)$ such that for t large enough, we have*

$$\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{ - y(C\beta)^{-1/\beta} \mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\right\}\right] \leqslant 2C_\beta(y) t^{-\rho} \ell\,(t)\,,$$

where ℓ is the slowly varying function at ∞ given in (4.11). *Further,*

$$\lim_{y\to\infty} C_\beta(e^y) = 0 \qquad \text{and} \qquad \lim_{y\to\infty} yC_\beta(e^y) = 0.$$

Proof. According to Patie and Savov [63, Theorem 2.20], we have that, under Spitzer's condition (**H1**), for any continuous and bounded function f on $\mathbb{R}^+$ and any constant $a \in (0, 1)$, we have

$$\lim_{t\to\infty} \frac{\widehat{\mathbb{E}}^{(e)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-a} f\Big(\mathtt{I}_{0,t}(-\beta\xi)\Big)\Big]}{\kappa(1/t, 0)} = \int_0^\infty f(x)\vartheta_a(\mathrm{d}x),$$

where ϑ_a is a positive measure on $(0,\infty)$. Now, let us define the continuous and bounded function $f(x) = x^a \exp(-y(C\beta)^{-1/\beta}x^{-1/\beta})$, thereby by the latter identity we deduce that, for any $a \in (0,1)$,

$$\lim_{t\to\infty} \frac{\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-y(C\beta)^{-1/\beta}\mathrm{I}_{0,t}(-\beta\xi)^{-1/\beta}\right\}\right]}{\kappa(1/t,0)} = C_\beta(y),$$

where

$$C_\beta(y) := \int_0^\infty x^a \exp\left\{-y(C\beta)^{-1/\beta}x^{-1/\beta}\right\}\vartheta_a(\mathrm{d}x). \tag{4.33}$$

On the other hand, since Spitzer's condition holds, we have that the Laplace exponent $\kappa(\cdot\,,0)$ of the ladder time process L^{-1} is regularly varying at 0^+ with index $\rho \in (0,1)$, i.e. from (4.7) we recall that,

$$\kappa(1/t,0) = t^{-\rho}\ell(t),$$

where ℓ is the slowly varying function at ∞ defined in (4.11). The latter implies that there exists $t_0 > 0$ such that if $t \geqslant t_0$

$$\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-y(C\beta)^{-1/\beta}\mathrm{I}_{0,t}(-\beta\xi)^{-1/\beta}\right\}\right] \leqslant 2C_\beta(y)t^{-\rho}\ell(t),$$

Furthermore, with the help of the Dominated Convergence Theorem, we obtain that

$$\lim_{y\to\infty} yC_\beta(e^y) = 2\int_0^\infty x^a \lim_{y\to\infty} y\exp\left\{-e^y(C\beta)^{-1/\beta}x^{-1/\beta}\right\}\vartheta_a(\mathrm{d}x) = 0.$$

Similarly, using Dominated Convergence Theorem, we have that $C_\beta(e^y) \to 0$ as $y \to \infty$. This concludes the proof. □

The following result makes precise the statement that only paths of the Lévy process with a very low running supremum give a substantial contribution to the speed of non-explosion.

Proposition 4.4.4. *Fix $z > 0$, $x < 0$ and $\epsilon \in (0,1)$. Suppose that assumptions* (**H1**) *and* (**A2**) *are satisfied. Then, we have*

$$\lim_{y\to\infty}\limsup_{t\to\infty} \frac{1}{\kappa(1/t,0)}\mathbb{P}_{(z,x)}\Big(Z_t < \infty,\ \overline{\xi}_{t-\epsilon} \geqslant y\Big) = 0. \tag{4.34}$$

Proof. Fix $z > 0,\ x < 0$ and $\epsilon \in (0,1)$. We begin by noting that condition (**A2**) allows us to find a lower bound for $v_t(0,0,\xi-\xi_0)$ in terms of the exponential functional of ξ.

Indeed, we observe from the backward differential equation given in (4.5) that

$$\frac{\partial}{\partial s}v_t(s,\lambda e^{-\xi_0},\xi-\xi_0)\leqslant Cv_t^{1+\beta}(s,\lambda e^{-\xi_0},\xi)e^{-\beta(\xi_s-\xi_0)},\quad v_t(t,\lambda e^{-\xi_0},\xi-\xi_0)=\lambda e^{-\xi_0}.$$

Integrating between 0 and t, we get

$$\frac{1}{v_t^{\beta}(0,\lambda e^{-\xi_0},\xi-\xi_0)}-\frac{1}{(\lambda e^{-\xi_0})^{\beta}}\geqslant C\beta\int_0^t e^{-\beta(\xi_s-\xi_0)}\mathrm{d}s,\quad C\beta>0.$$

Now, letting $\lambda\downarrow 0$ and taking into account that $\beta\in(-1,0)$ and $C<0$, we deduce the following inequality for all $t\geqslant 0$,

$$v_t(0,0,\xi-\xi_0)\geqslant\Big(C\beta\mathtt{I}_{0,t}(\beta(\xi-\xi_0))\Big)^{-1/\beta},\tag{4.35}$$

where $\mathtt{I}_{0,t}(\beta(\xi-\xi_0))$ is the exponential functional of the Lévy process $\beta(\xi-\xi_0)$ see for instance (3.29). Hence, the quenched non-explosion probability given in (4.18) may be bounded in terms of this functional. That is to say, for all $t>0$,

$$\mathbb{P}_{(z,x)}\Big(Z_t<\infty\;\Big|\;\xi\Big)=\exp\Big\{-zv_t(0,0,\xi-\xi_0)\Big\}\leqslant\exp\bigg\{-z\Big(C\beta\mathtt{I}_{0,t}(\beta(\xi-\xi_0))\Big)^{-1/\beta}\bigg\}.$$

Therefore, conditioning on the environment and using the notation $\widehat{\mathbb{P}}^{(e)}$ for the law of the dual process $\widehat{\xi}$, we obtain that, for any $y>x$

$$\begin{aligned}\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}\geqslant y\Big)&=\mathbb{E}_x^{(e)}\Big[\mathbb{P}_{(z,x)}\Big(Z_t<\infty\;\Big|\;\xi\Big)\mathbf{1}_{\{\overline{\xi}_{t-\epsilon}\geqslant y\}}\Big]\\&\leqslant\widehat{\mathbb{E}}^{(e)}\Big[\exp\Big\{-z(C\beta)^{-1/\beta}\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\Big\}\mathbf{1}_{\{\underline{\xi}_{t-\epsilon}\leqslant x-y\}}\Big].\end{aligned}\tag{4.36}$$

Let $w=x-y$ and $t_0>0$. Now, we split the event $\{\tau_w^-\leqslant t-\epsilon\}$ for $3t_0<t$ and $0<\epsilon<1$, as follows

$$\{\tau_w^-\leqslant t-\epsilon\}=\{0<\tau_w^-\leqslant(t-t_0)/2\}\cup\{(t-t_0)/2<\tau_w^-\leqslant t-\epsilon\}.$$

By the monotonicity of the mapping $t\mapsto\mathtt{I}_{0,t}(-\beta\xi)$, we have, under the event $\{0<\tau_w^-\leqslant(t-t_0)/2\}$, that

$$t_0<\tau_w^-<\tau_w^-+\frac{t+t_0}{2}\leqslant t\qquad\text{and}\qquad\mathtt{I}_{0,t}(-\beta\xi)\geqslant\mathtt{I}_{\tau_w^-,\tau_w^-+\frac{t+t_0}{2}}(-\beta\xi).$$

Similarly, under the event $\{(t-t_0)/2<\tau_w^-\leqslant t-\epsilon\}$, we obtain

$$\frac{t-t_0}{2}<\tau_w^-<\tau_w^-+\epsilon\leqslant t\qquad\text{and}\qquad\mathtt{I}_{0,t}(-\beta\xi)\geqslant\mathtt{I}_{\tau_w^-,\tau_w^-+\epsilon}(-\beta\xi).$$

Next, appealing to the strong Markov property of ξ, we deduce

$$\begin{aligned}
&\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-z(C\beta)^{-1/\beta}\mathtt{I}_{\tau_w^-,\tau_w^-+\frac{t+t_0}{2}}(-\beta\xi)^{-1/\beta}\right\};\ 0<\tau_w^-\leqslant(t-t_0)/2\right]\\
&\quad\leq\quad \widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-ze^{-w}(C\beta)^{-1/\beta}\left(\int_0^{\frac{t+t_0}{2}}e^{\beta\left(\xi_{s+\tau_w^-}-\xi_{\tau_w^-}\right)}\mathrm{d}s\right)^{-1/\beta}\right\};\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.0<\tau_w^-\leqslant(t-t_0)/2\right]\\
&\quad=\quad \widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-ze^{-w}(C\beta)^{-1/\beta}\mathtt{I}_{0,\frac{t+t_0}{2}}(-\beta\xi)^{-1/\beta}\right\}\right]\widehat{\mathbb{P}}^{(e)}\left(0<\tau_w^-\leqslant\frac{t-t_0}{2}\right).
\end{aligned}$$

Now from Lemma 4.4.3, for t sufficiently large, we have

$$\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-ze^{-w}(C\beta)^{-1/\beta}\mathtt{I}_{0,\frac{t+t_0}{2}}(-\beta\xi)^{-1/\beta}\right\}\right]\leqslant 2C_\beta(ze^{-w})\left(\frac{t+t_0}{2}\right)^{-\rho}\ell\left(\frac{t+t_0}{2}\right),$$

where ℓ is the slowly varying function at ∞ given in (4.11) and $C_\beta(ze^{-w})$ is the function defined in (4.33). Therefore, for t large enough, we get

$$\begin{aligned}
&\widehat{\mathbb{E}}^{(e)}\left[\exp\left\{-z(C\beta)^{-1/\beta}\mathtt{I}_{\tau_w^-,\tau_w^-+\frac{t+t_0}{2}}(-\beta\xi)^{-1/\beta}\right\};\ 0<\tau_w^-\leqslant(t-t_0)/2\right]\\
&\qquad\qquad\qquad\qquad\qquad\leqslant 2C_\beta(ze^{-w})\left(\frac{t+t_0}{2}\right)^{-\rho}\ell\left(\frac{t+t_0}{2}\right).
\end{aligned}$$

Using the same arguments as above and Lemmas 4.4.2 and 4.4.3, we obtain the following sequence of inequalities for t sufficiently large,

$$\begin{aligned}
&\widehat{\mathbb{E}}^{(e)}\Big[\exp\Big\{-z(C\beta)^{-1/\beta}\mathtt{I}_{\tau_w^-,\tau_w^-+\epsilon}(-\beta\xi)^{-1/\beta}\Big\};\ (t-t_0)/2<\tau_w^-\leqslant t-\epsilon\Big]\\
&\quad\leqslant\widehat{\mathbb{E}}^{(e)}\Big[\exp\Big\{-ze^{-w}(C\beta)^{-1/\beta}\mathtt{I}_{0,\epsilon}(-\beta\xi)^{-1/\beta}\Big\}\Big]\widehat{\mathbb{P}}^{(e)}\left(\frac{t-t_0}{2}<\tau_w^-\leqslant t-\epsilon\right)\\
&\quad\leqslant 2C_\beta(ze^{-w})\epsilon^{-\rho}\ell(\epsilon)C_3\left(C_4\left(\frac{2(t-\epsilon)}{t-t_0}\right)^{\eta+\rho}-1\right)U(-w)\,(t-\epsilon)^{-\rho}\,\ell\,(t-\epsilon)\\
&\quad\leqslant 2C_\beta(ze^{-w})\epsilon^{-\rho}\ell(\epsilon)C_3\left(C_4 2^{\eta+\rho}\left(1+\frac{t_0-\epsilon}{2t_0}\right)^{\eta+\rho}-1\right)U(-w)\,(t-\epsilon)^{-\rho}\,\ell\,(t-\epsilon),
\end{aligned}$$

where $\eta>0$.

Hence plugging this back into (4.36) (and similarly as in the proof of Lemma 4.4

in [52]), we get, for t large enough,

$$\begin{aligned}\widehat{\mathbb{E}}^{(e)}\Big[\exp\Big\{-z(C\beta)^{-1/\beta}\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\Big\};\ \tau_w^-\leqslant t-\epsilon\Big]\\ \leqslant 2C_\beta(ze^{-w})\left(\frac{t+t_0}{2}\right)^{-\rho}\ell\left(\frac{t+t_0}{2}\right)+2C_\beta(ze^{-w})\epsilon^{-\rho}\ell(\epsilon)C_3\\ \times\left(C_4 2^{\eta+\rho}\left(1+\frac{t_0-\epsilon}{2t_0}\right)^{\eta+\rho}-1\right)U(-w)\,(t-\epsilon)^{-\rho}\,\ell\,(t-\epsilon)\,.\end{aligned}$$

Define the following constant:

$$C_5(ze^{-w})=2C_\beta(ze^{-w})\vee 2C_\beta(ze^{-w})\epsilon^{-\rho}\ell(\epsilon)C_3\left(C_4 2^{\eta+\rho}\left(1+\frac{t_0-\epsilon}{2t_0}\right)^{\eta+\rho}-1\right),$$

and recalling the definition of the constant $C_\beta(ze^{-w})$ and Lemma 4.4.3, we have that

$$\lim_{y\to\infty}C_\beta(ze^{y-x})=0\qquad\text{and}\qquad\lim_{y\to\infty}yC_\beta(ze^{y-x})=0.$$

Now, appealing to Potter's Theorem, we deduce, for any $A_1,A_2>1$ and $\delta_1,\delta_2>0$, that for t large enough,

$$\begin{aligned}\frac{1}{\kappa(1/t,0)}\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}\geqslant y\Big)&=\frac{t^\rho}{\ell(t)}\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}\geqslant y\Big)\\ &=C_5(ze^{-w})\left\{\left(\frac{2t}{t+t_0}\right)^\rho\frac{\ell((t+t_0)/2)}{\ell(t)}+U(-w)\left(\frac{t}{t-\epsilon}\right)^\rho\frac{\ell(t-\epsilon)}{\ell(t)}\right\}\\ &\leqslant C_5(ze^{y-x})\left\{A_1\left(\frac{2t}{t+t_0}\right)^{\rho-\delta_1}+U(y-x)A_2\left(\frac{t}{t-\epsilon}\right)^{\rho-\delta_2}\right\}.\end{aligned}$$

Finally, taking into account that the renewal function U grows at most linearly, i.e., $U(y-x)=\mathcal{O}(y-x)$, and letting $t\to\infty$ and then $y\to\infty$ we obtain the desired result. □

With the previous Propositions 4.4.1 and 4.4.4 in hand, we may now proceed to the proof of Theorem 4.1.3.

Proof of Theorem 4.1.3. Fix $\varsigma,z>0$, $x<0$ and $\epsilon\in(0,1)$. We begin by noting, from Proposition 4.4.4, that we may choose $y>0$ such that for t large enough,

$$\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}\geqslant y\Big)\leqslant\varsigma\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}<y\Big).\tag{4.37}$$

Now, note that for t large enough, we get $\{Z_t<\infty\}\subset\{Z_{t-\epsilon}<\infty\}$ and using the previous inequality, it follows,

$$\begin{aligned}\mathbb{P}_z(Z_t<\infty) &= \mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}\geqslant y\Big)+\mathbb{P}_{(z,x)}\Big(Z_t<\infty,\ \overline{\xi}_{t-\epsilon}<y\Big)\\ &\leqslant (1+\varsigma)\mathbb{P}_{(z,x-y)}\Big(Z_{t-\epsilon}<\infty,\ \overline{\xi}_{t-\epsilon}<0\Big).\end{aligned}$$

In other words, for every $\varsigma>0$ there exists $y'<0$ such that for t large enough

$$\begin{aligned}(1-\varsigma)\frac{\mathbb{P}_{(z,y')}\Big(Z_t<\infty,\ \overline{\xi}_t<0\Big)}{\kappa(1/t,0)} &\leqslant \frac{\mathbb{P}_z(Z_t<\infty)}{\kappa(1/t,0)}\\ &\leqslant (1+\varsigma)\frac{\mathbb{P}_{(z,y')}\Big(Z_{t-\epsilon}<\infty,\ \overline{\xi}_{t-\epsilon}<0\Big)}{\kappa(1/(t-\epsilon),0)}\frac{\kappa(1/(t-\epsilon),0)}{\kappa(1/t,0)}.\end{aligned}$$

Next, using that the function $\kappa(\cdot,0)$ is regularly varying at 0, and then appealing to Potter's Theorem in Bingham et al. [15], we see that, for any $A>1$ and $\eta>0$,

$$\lim_{t\to\infty}\frac{\kappa(1/(t-\epsilon),0)}{\kappa(1/t,0)}=\lim_{t\to\infty}\frac{\ell(t-\epsilon)}{\ell(t)}\left(\frac{t-\epsilon}{t}\right)^{-\rho}\leqslant\lim_{t\to\infty}A\left(\frac{t}{t-\epsilon}\right)^{\rho+\eta}=A.$$

On the other hand, according to Proposition 4.4.1, there exists $0<c(z,y')<\infty$ such that

$$\lim_{t\to\infty}\frac{1}{\kappa(1/t,0)}\mathbb{P}_{(z,y')}\Big(Z_t<\infty,\ \overline{\xi}_t<0\Big)=c(z,y')U(-y').$$

Hence, as a consequence of the above facts, we get

$$(1-\varsigma)c(z,y')U(-y')\leqslant\lim_{t\to\infty}\frac{\mathbb{P}_z(Z_t<\infty)}{\kappa(1/t,0)}\leqslant(1+\varsigma)c(z,y')U(-y')A.$$

We observe that y' is a sequence which may depend on ς and z. Further, this sequence y' goes to minus infinity as ς goes to 0. Thus, for any sequence $y_\varsigma(z)$, we have

$$\begin{aligned}0<(1-\varsigma)c(z,y_\varsigma(z))U(-y_\varsigma(z)) &\leqslant\lim_{t\to\infty}\frac{\mathbb{P}_z(Z_t<\infty)}{\kappa(1/t,0)}\\ &\leqslant(1+\varsigma)c(z,y_\varsigma(z))U(-y_\varsigma(z))A<\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}0<\liminf_{\varsigma\to0}(1-\varsigma)c(z,y_\varsigma(z))U(-y_\varsigma(z)) &\leqslant\lim_{t\to\infty}\frac{\mathbb{P}_z(Z_t<\infty)}{\kappa(1/t,0)}\\ &\leqslant\limsup_{\varsigma\to0}(1+\varsigma)c(z,y_\varsigma(z))U(-y_\varsigma(z))A<\infty.\end{aligned}$$

Since A can be taken arbitrary close to 1, we deduce

$$0 < \lim_{t\to\infty} \frac{\mathbb{P}_z(Z_t < \infty)}{\kappa(1/t, 0)} = \mathfrak{C}_1(z) := \lim_{\varsigma\to 0} c(z, y_\varsigma(z))U(-y_\varsigma(z)) < \infty,$$

which completes the proof. □

4.5 Subcritical-explosion regime

This section is devoted to the proof of Theorem 4.1.4. The proof follows similar ideas as those used in the proof of Proposition 3 in Palau and Pardo [57].

Proof. Let $z > 0$ and $x \in \mathbb{R}$. We begin by recalling, from Lemma 4.3.2, that we can deduce the following identity,

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t < \infty) = \lim_{t\to\infty} \mathbb{E}_x^{(e)}\Big[\exp\{-zv_t(0, 0, \xi - \xi_0)\}\Big],$$

we observe that the left-hand side of the above equation does not depend of the initial value x of the Lévy process ξ.

Hence, as soon as we can establish that

$$\lim_{t\to\infty} v_t(0, 0, \xi - \xi_0) < \infty, \quad \mathbb{P}_x^{(e)} - \text{a.s.},$$

our proof is completed. Now, from the proof of Lemma 4.3.3, we have the following inequality

$$\lim_{t\to\infty} v_t(0, 0, \xi - \xi_0) \leqslant \lambda e^{-\xi_0} \exp\left\{\int_0^\infty \widehat{\Phi}_\lambda(\widehat{\xi}_s)\mathrm{d}s\right\}, \tag{4.38}$$

where $\widehat{\xi} = -\xi$ denotes the dual process and $\widehat{\Phi}_\lambda(\cdot)$ is the function defined in (4.12). We recall that in this regime the dual process $\widehat{\xi}$ drifts to ∞, $\mathbb{P}_x^{(e)}$ -a.s. Thus, in order to prove that the integral in (4.38) is finite, let us introduce $\varsigma = \sup\{t \geqslant 0 : \widehat{\xi}_t \leqslant 0\}$ and observe that

$$\int_0^\infty \widehat{\Phi}_\lambda(\widehat{\xi}_s)\mathrm{d}s = \int_0^\varsigma \widehat{\Phi}_\lambda(\widehat{\xi}_s)\mathrm{d}s \;+\; \int_\varsigma^\infty \widehat{\Phi}_\lambda(\widehat{\xi}_s)\mathrm{d}s.$$

Since $\varsigma < \infty$, $\mathbb{P}_x^{(e)}$ -a.s., it follows that the first integral in the right-hand side above is finite $\mathbb{P}_x^{(e)}$-a.s. For the second integral, we may appeal to Theorem 1 in Erickson and Maller [29], which guarantees, under condition (4.14), that

$$\int_\varsigma^\infty \widehat{\Phi}_\lambda(\widehat{\xi}_s)\mathrm{d}s < \infty, \qquad \mathbb{P}_x^{(e)} - \text{a.s.}$$

Furthermore, if $\mathbb{E}^{(e)}\Big[\widehat{\xi}_1\Big] < \infty$, then $\lim\limits_{x\to\infty} A_\xi(x)$ is finite. In particular, it follows that the integral condition (4.14) is equivalent to

$$\int_0^\infty \widehat{\Phi}_\lambda(u)\mathrm{d}u < \infty.$$

Moreover, we have

$$\int_0^\infty \widehat{\Phi}_\lambda(u)\mathrm{d}u = \int_0^\infty \int_{(0,\infty)} \exp\{-\lambda e^u y\}\bar{\mu}(y)\mathrm{d}y\mathrm{d}u = \int_0^\infty \int_1^\infty \frac{\exp\{-\lambda y w\}}{w}\mathrm{d}w\bar{\mu}(y)\mathrm{d}y,$$

Now by the definition of the exponential integral given in (4.13), we deduce that condition (4.14) is equivalent to

$$\int_0^\infty \mathtt{E}_1(\lambda y)\bar{\mu}(y)\mathrm{d}y < \infty,$$

which concludes the proof. □

4.6 Future research

In this section, we establish some conjectures regarding to the probability of non-explosion for CSBPs in a Lévy environment that we are currently studying. We would like to obtain these results as they would give us a complete characterisation of the explosion event in our setting.

In Theorem 4.1.1 we characterised the law of $(Z_t e^{-\xi_t}, t \geqslant 0)$ via a backward differential equation when the branching mechanism is given by the negative of a subordinator and with infinite mean. Nevertheless, we would like to obtain such result also for the case when the branching mechanism ψ is given by the Laplace exponent of a spectrally negative Lévy process with infinite mean. The intuition we have about the veracity of such statement lies in the so-called Neveu case. Recall that the Neveu branching process in a Lévy random environment has branching mechanism given by

$$\psi(\lambda) = \lambda \log \lambda = c\lambda + \int_{(0,\infty)} \Big(e^{-\lambda y} - 1 + \lambda y \mathbf{1}_{\{y<1\}}\Big) y^{-2}\mathrm{d}y, \qquad y > 0,$$

where $c \in \mathbb{R}$ is a suitable constant. Note that, in this case we have $\psi'(0+) = -\infty$. Additionally, the backward differential equation (4.5),

$$\frac{\partial}{\partial s} v_t(s,\lambda,\xi) = v_t(s,\lambda,\xi) \log\Big(e^{-\xi_s} v_t(s,\lambda,\xi)\Big), \qquad s \in [0,t], \qquad v_t(t,\lambda,\xi) = \lambda,$$

has a solution which is given by

$$v_t(s,\lambda,\xi) = \exp\left\{e^s\left(\int_s^t e^{-u}\xi_u \mathrm{d}u + e^{-t}\log\lambda\right)\right\},$$

(see Palau and Pardo [58] for further details). Therefore, we believe that the following conjecture holds.

Conjecture 4.6.1. *Assume that ψ is the Laplace exponent of a spectrally negative Lévy process such that $\psi'(0+) = -\infty$. For every $z, \lambda, t > 0$ and $x \in \mathbb{R}$, we have*

$$\mathbb{E}_{(z,x)}\left[\exp\left\{-\lambda Z_t e^{-\xi_t}\right\}\Big|\,\xi\right] = \exp\left\{-z v_t(0, \lambda e^{-\xi_0}, \xi - \xi_0)\right\},$$

where for any $\lambda, t \geqslant 0$, the function $v_t : s \in [0,t] \to v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0)$ is an a.s. solution of the backward differential equation

$$\frac{\partial}{\partial s} v_t(s,\lambda,\xi) = e^{\xi_s}\psi\Big(v_t(s,\lambda,\xi)e^{-\xi_s}\Big), \qquad a.e. \qquad s \in [0,t]$$

and with terminal condition $v_t(t,\lambda,\xi) = \lambda$.

In this case the Carathéodory Existence Theorem 4.2.1 is not directly applicable. One main difficulty in applying this result is having a control over the growth of the function $f(s,\theta) = e^{\xi_s}\psi(\theta e^{-\xi_s})$. To overcome this obstacle, we will consider a stronger version of Theorem 4.2.1 (see [65, Section 3]) and also the following identity, which follows from the Wiener–Hopf factorisation,

$$\psi(\theta) = (\theta - \alpha)\Lambda(\theta), \quad \text{for all} \quad \theta \geqslant 0, \tag{4.39}$$

where Λ is the Laplace exponent of a subordinator, $\alpha = \widehat{\psi}(0) > 0$ with $\widehat{\psi}$ the right inverse of the function ψ.

Once we are able to prove Conjecture 4.6.1, we will have a complete criterion regarding to the conservativeness of the CSBPs in a Lévy environment in terms of the branching mechanism. In other words, we state that

Assuming that Conjecture 4.6.1 holds and that ψ is the Laplace exponent of a spectrally negative Lévy process with $\psi'(0+) = -\infty$, a continuous-state branching process in a Lévy environment is conservative if and only if

$$\int_{0+} \frac{1}{|\psi(z)|}\mathrm{d}z = \infty.$$

In fact, this result can be proved following the same strategy used in Proposition 4.1.2 together with identity (4.39). Finally, we state the following conjecture concerning the speed of the non-explosion probability for a CSBPs in a supercritical-explosion regime.

Conjecture 4.6.2 (Supercritical-explosion regime)**.** *Assume that* $\Phi'_\xi(0+) > 0$. *Suppose that* ψ *is the negative of a subordinator, that condition* (**A2**) *holds and some others technical conditions on the characteristics of the Lévy process* ξ. *Then, for any* $z > 0$, *there exists* $0 < \mathfrak{C}_3(z) < \infty$ *such that*

$$\lim_{t\to\infty} t^{3/2} e^{-t\Phi_\xi(\gamma)} \mathbb{P}_z(Z_t < \infty) = \mathfrak{C}_3(z),$$

where γ *is such that* $\Phi'_\xi(\gamma) = 0$.

Roughly speaking, our aim is to study the event of non-explosion at time t in two different situations that depend on the behaviour of the supremum of the environment. To be more precise, we split the event of non-explosion as follows:

$$\mathbb{P}_z(Z_t < \infty) = \mathbb{P}_{(z,x)}\Big(Z_t < \infty,\ \overline{\xi}_t \geqslant 0\Big) + \mathbb{P}_{(z,x)}\Big(Z_t < \infty,\ \overline{\xi}_t < 0\Big), \tag{4.40}$$

for $z > 0$ and $x < 0$. For the asymptotic behaviour of the first probability in the right-hand side above, it will be necessary to use condition (**A2**) which give us the following lower bound

$$v_t(0, 0, \xi - \xi_0) \geqslant \Big(C\beta \mathtt{I}_{0,t}(\beta(\xi - \xi_0))\Big)^{-1/\beta},$$

With this in hand, it will be possible to handle the expectation

$$\mathbb{E}^{(e)}_x\Big[\mathbb{P}_{(z,x)}\Big(Z_t < \infty \ \Big|\ \xi\Big)\mathbf{1}_{\{\overline{\xi}_t \geqslant 0\}}\Big],$$

in order to obtain its long-term behaviour. On the other hand, for the second probability in the right-hand of (4.40), a more elaborate argument will be needed. To this end, we write

$$\mathbb{P}_{(z,x)}\Big(Z_t < \infty,\ \overline{\xi}_t < 0\Big) \ = \ \mathbb{P}_{(z,x)}\Big(Z_t < \infty \ \Big|\ \overline{\xi}_t < 0\Big)\mathbb{P}^{(e)}_x\Big(\overline{\xi}_t < 0\Big),$$

and we will study the conditional probability using the fluctuation theory of Lévy processes and the approach developed in Chapter 5 for the non-extinction probability in the intermediate subcritical regime.

Chapter 5

Extinction rates of CSBPs in a subcritical Lévy environment

The aim of this chapter is to study the speed of the non-extinction probability for CSBPs in a subcritical Lévy environment for a more general class of branching mechanisms than the stable class already discussed in Section 3.6. In this regime, the underlying Lévy process drifts to $-\infty$. Further, as it was observed in [52] and [59], there is another phase transition in such regime which depends on whether $\Phi'_\xi(1)$ is less, equal or greater than 0. These regimes are known in the literature as: *strongly*, *intermediate* and *weakly subcritical regime*, respectively (see e.g. Theorem 3.6.1). Here, we study the exact asymptotic behaviour of the non-extinction probability in the intermediate and strongly sub-regimes, under certain assumptions on the Lévy process associated to the environment and the branching mechanism. For our purpose, we combine the approach developed in [2, 33], for the discrete time setting, with the fluctuation theory of Lévy processes. A similar strategy has been developed in [7] to study the extinction rate for CSBPs in a critical Lévy environment. The chapter is structured as follows. In Section 5.1 we state our main results. Sections 5.2 and 5.3 are devoted to the study of the speed of non-extinction probability for CSBPs in the strongly and intermediate subcritical regimes, respectively.

5.1 Main results

Let $Z = (Z_t, t \geqslant 0)$ be the CSBP in the Lévy environment $(S_t, t \geqslant 0)$ defined in Chapter 3. Before we proceed with the main results of this chapter, let us first recall some notation and some properties of the functional $v_t(s, \lambda, \xi)$ for $0 \leqslant s \leqslant t$ already discussed in Chapter 3.

We denote by $h_{s,t}$ the random semigroup $h_{s,t}(\lambda) = e^{-\xi_s} v_t(s, \lambda e^{\xi_t}, \xi)$ for all $s \in [0,t]$ and $\lambda \geqslant 0$. According to [38, Section 2], the mapping $s \mapsto h_{s,t}(\lambda)$ is the unique positive pathwise solution of (3.9). For our purposes, it is necessary to extend the functional $(v_t(s,\lambda,\xi), s \in [0,t])$ to $[-t, 0]$, this was already done in He et al. [38] in order to study CSBPs in a Lévy environment with immigration. The latter will appear implicitly in our arguments.

In order to introduce the extension of the functional $v_t(s,\lambda,\xi)$ to $s \leq 0$ that appears in He et al. [38, Section 5], we first define a homogeneous Lévy process in $\mathbb{R}$. Let us consider an independent copy $(\xi'_t, t \geqslant 0)$ of the Lévy process $(\xi_t, t \geqslant 0)$, thus $\Xi = (\Xi_t, -\infty < t < \infty)$ the time homogeneous Lévy process indexed by $\mathbb{R}$ is defined as follows: $\Xi_0 = \xi_0 = 0$ and

$$\Xi_t = -\lim_{s \downarrow -t} \xi'_s \quad \text{for} \quad t < 0 \qquad \text{and} \qquad \Xi_t = \xi_t \quad \text{for} \quad t > 0 \tag{5.1}$$

Note that the latter definition ensures that the Lévy process Ξ has càdlàg paths on $(-\infty, \infty)$. Next, we use the definition of Ξ to naturally extend the backward differential equation (3.5) on $s \leqslant 0$. In other words, the mapping $s \mapsto v_0(s,\lambda,\Xi)$ is the unique positive pathwise solution of

$$v_0(s,\lambda,\Xi) = \lambda - \int_s^0 e^{\Xi_r} \psi_0\Big(e^{-\Xi_r} v_0(r,\lambda,\Xi)\Big) \mathrm{d}r, \qquad s \leqslant 0. \tag{5.2}$$

Implicitly, it also follows that for $s \leqslant 0$ the function $s \mapsto h_{s,0}(\lambda) = e^{-\Xi_s} v_0(s, \lambda e^{\Xi_0}, \Xi)$ is the unique positive pathwise solution to the equation

$$h_{s,0}(\lambda) = e^{-\Xi_s}\lambda - \int_s^0 e^{\Xi_r - \Xi_s} \psi_0\Big(h_{r,0}(\lambda)\Big) \mathrm{d}r, \qquad s \leqslant 0. \tag{5.3}$$

For further details of this extension we refer to He et al. [38, Section 5].

We now state our main results which are devoted to the speed of the non-extinction probability for CSBPs in a subcritical Lévy environment. In this regime, the probability of non-extinction at time t decays at an exponential rate contrary to the critical case (see Theorem 3.6.2) where the rates are of the polynomial type. Such behaviour is inherited from the Esscher change of measure defined in (3.19). In order to introduce our main results we require some technical assumptions on the branching mechanism and on the environment which will control the event of survival. The reader will see below that these assumptions are slightly different between regimes and are in accordance with those already assumed in the literature.

In the strongly subcritical regime, we make two assumptions: one on the branching mechanism and the other on the environment. Our first assumption is the so-called $x \log x$ moment condition for the branching mechanism. To be more precise, we assume that the Lévy measure μ fulfils the following moment condition

$$\int_1^\infty u \log(u)\mu(\mathrm{d}u) < \infty. \tag{\textbf{H2*}}$$

We point out here that the latter condition is also needed in the discrete setting to study the long-term behaviour of branching processes in a strongly subcritical random environment (see e.g. [4, Theorem 1.1]).

For this regime, we show that the non-extinction probability at time t decays at the same rate as the expected generation size, i.e., as $\mathbb{E}_z[Z_t]$ up to a multiplicative constant. Note that this behaviour is similar to that of subcritical Galton-Watson processes as well as discrete branching processes in a random environment in the strong subcritical regime. It is also worth mentioning here that from (3.4), we have

$$\mathbb{E}_z[Z_t] = z\mathbb{E}[e^{\xi_t}] = ze^{\Phi_\xi(1)t}.$$

In other words, in this regime the non-extinction probability decays in an exponential rate up to a multiplicative constant which is proportional to the initial state of the population. In addition, observe that here we obtain a very general result since we do not require condition (**H3**) on the branching mechanism. We now state our second main result.

Theorema 5.1.1 (Strongly subcritical regime)**.** *Suppose that conditions* (3.26)*,* (**H2***)*,* (3.18) *with* $\vartheta^+ = 1$*,* $\Phi'_\xi(0) < 0$ *and* $\Phi'_\xi(1) < 0$*. We also assume*

$$\int_0^\infty \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty.$$

Then for every $z > 0$*, we have*

$$\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z\mathfrak{B}_2,$$

where

$$\mathfrak{B}_2 = \mathbb{E}^{(e,1)}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\mathrm{d}\lambda\right] \in (0,\infty).$$

In general, it seems difficult to compute explicitly the constant $\mathfrak{B}_2$. Nevertheless, in the stable case the constant $\mathfrak{B}_2$ can be computed explicitly and coincides with the constant that appears in [52, Theorem 5.1]. In other words, for the stable branching mechanism (3.27) with $\beta \in (0,1)$ and $C > 0$, we have

$$\mathfrak{B}_2 = (\beta C)^{-1/\beta} \mathbb{E}^{(e,1)} \left[\left(\int_0^\infty e^{\beta \xi_u} \mathrm{d}u \right)^{-1/\beta} \right],$$

(see the discussion at the end of Section 5.2 for further details).

Our second main result deals with the intermediate subcritical regime. Here, we assume that the branching mechanism satisfies condition (**H3**) and that the Lévy measure μ satisfies the $x \log x$ moment condition (**H2***). In addition, our arguments require the existence of some exponential moments of the underlying Lévy process ξ. More precisely, we assume that condition (3.18) holds with $\vartheta^- = 0$ and $\vartheta^+ > 1$. Observe that this condition together with $\Phi'_\xi(1) = 0$ imply Spitzer's condition (**H1**) under the measure $\mathbb{P}^{(e,1)}$ with $\rho = 1/2$. Nonetheless, we believe that it is possible to obtain the result without this exponential moment assumption but up to now we do not know how to control the behaviour of

$$e^{-\xi_t} \mathbb{P}_{(z,x)} \Big(Z_t > 0 \;\Big|\; \xi \Big),$$

as t increases. Nevertheless, the latter random variable can be controlled under the exponential moment assumption together with the upper bound of the branching mechanism with the stable case.

We believe that, without the exponential moment condition (3.18), the survival probability at time t must decays in an exponential rate with a factor of order $t^{-\rho}\ell(t)$ where ℓ is a slowly varying function at ∞ and ρ is the index that appears in Sptizer's condition under $\mathbb{P}^{(e,1)}$.

The following result provides the asymptotic behaviour of the probability of non-extinction in the intermediate subcritical regime.

Theorema 5.1.2 (Intermediate subcritical regime)**.** *Suppose that conditions* (**H2***) *and* (**H3**) *holds. We also assume that the exponential moment condition* (3.18) *holds with* $\vartheta^- = 0$ *and* $\vartheta^+ > 1$ *and moreover that* $\Phi'_\xi(0) < 0$ *and* $\Phi'_\xi(1) = 0$. *Finally, we also require that for* $x < 0$

$$\int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow} \left[\exp\left\{ -\int_{-1}^0 \psi_0'(h_{s,0}(\lambda)) \mathrm{d}s \right\} \right] \mathrm{d}\lambda < \infty. \tag{5.4}$$

Then for every $z > 0$, *we have*

$$\lim_{t\to\infty} t^{3/2} e^{-\Phi_\xi(1)t} \mathbb{P}_z(Z_t > 0) = z \mathbb{E}^{(e,1)} \Big[H_1 \Big] \sqrt{\frac{2}{\pi \Phi''_\xi(1)}} \mathfrak{B}_3,$$

where

$$\mathfrak{B}_3 = \lim_{x\to-\infty} U^{(1)}(-x)\mathbb{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathrm{d}\lambda\right].$$

We conclude this section with the following conjecture concerning the speed of the non-extinction probability for a CSBPs in a weakly subcritical regime.

Conjecture 5.1.1 (Weakly subcritical regime)**.** *Suppose that conditions* (**H2**)- (**H3**) *hold. We also assume that the Laplace exponent of ξ satisfies $\Phi_\xi'(0) < 0 < \Phi_\xi'(1)$ and that there exist $\gamma \in (0,1)$ which solves $\Phi_\xi'(\gamma) = 0$. Then for any $z > 0$, there exists $0 < \mathfrak{B}_1(z) < \infty$ such that*

$$\lim_{t\to\infty} t^{-3/2} e^{-\Phi_\xi(\gamma)t}\mathbb{P}_z(Z_t > 0) = \mathfrak{B}_1(z).$$

In section 5.4, we make a brief comment about the strategy that we believe should be followed to obtain this result.

5.2 Strongly subcritical regime

This section is devoted to the proof of Theorem 5.1.1. In other words, we study the speed of the non-extinction probability for CSBPs in a strongly subcritical Lévy environment. In this regime, the Lévy process ξ fulfils the conditions $\Phi_\xi'(0) < 0$ and $\Phi_\xi'(1) < 0$. Note that the previous conditions implies that ξ drifts to $-\infty$ under $\mathbb{P}^{(e)}$ and also under the Esscher change of measure $\mathbb{P}^{(e,1)}$ defined in (3.19). As we mentioned before, for our arguments in this section we will require the extended homogeneous Lévy process on $\mathbb{R}$, defined by $\Xi = (\Xi_t, -\infty < t < \infty)$ in (5.1).

Proof of Theorem 5.1.1. Let $z > 0$. We begin by noting that, conditioning on the environment and then using the Esscher change of measure given in (3.19), we have

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) &= e^{-\Phi_\xi(1)t}\mathbb{E}^{(e)}\left[e^{-\xi_t}e^{\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \;\Big|\; \xi\Big)\right] \\ &= \mathbb{E}^{(e,1)}\left[e^{-\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \;\Big|\; \xi\Big)\right]. \end{aligned}$$

Recall from (3.8) in Chapter 3, that for any $\lambda \geqslant 0$ and $t \geqslant 0$ the cumulant random semigroup $h_{0,t}(\lambda) = e^{-\xi_0}v_t(s, \lambda e^{\xi_t}, \xi)$ satisfies

$$\mathbb{E}_{(z,0)}\left[e^{-\lambda Z_t} \;\Big|\; \xi\right] = \exp\{-z h_{0,t}(\lambda)\}.$$

Denote for each fixed $t \geqslant 0$,

$$G_t(\lambda) = e^{-\xi_t}\Big(1 - \exp\{-zh_{0,t}(\lambda)\}\Big), \qquad \text{for} \quad \lambda \geqslant 0.$$

Observe that the quenched survival probability of the process $(Z_t, t \geqslant 0)$ is given by $\mathbb{P}_{(z,0)}\Big(Z_t > 0 \,\Big|\, \xi\Big) = 1 - \exp\{-zh_{0,t}(\infty)\}$. Then

$$G_t(0) = 0 \qquad \text{and} \qquad G_t(\infty) = e^{-\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \,\Big|\, \xi\Big).$$

Since the mapping $\lambda \mapsto h_{0,t}(\lambda)$ is differentiable, then so does $G_t(\cdot)$. In view of the above arguments, we deduce

$$e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = \mathbb{E}^{(e,1)}\Big[G_t(\infty)\Big] = \mathbb{E}^{(e,1)}\left[\int_0^\infty G_t'(\lambda)\mathrm{d}\lambda\right] = \int_0^\infty \mathbb{E}^{(e,1)}\Big[G_t'(\lambda)\Big]\mathrm{d}\lambda, \tag{5.5}$$

where in the last equality, the expectation and the integral may be exchanged using Fubini's Theorem. Next, we take the limit as $t \to \infty$ in the above equality and we will appeal to the Dominated Convergence Theorem in order to interchange the limit with the integral on the right-hand side. With this purpose in mind, we need to find a function $g(\lambda)$ such that $\mathbb{E}^{(e,1)}\Big[|G_t'(\lambda)|\Big] \leqslant g(\lambda)$ for all $t \geqslant 1$ and

$$\int_0^\infty g(\lambda)\mathrm{d}\lambda < \infty. \tag{5.6}$$

First, we analyse the expectation $\mathbb{E}^{(e,1)}\Big[|G_t'(\lambda)|\Big]$. Note that from the definition of the function $G_t(\lambda)$, we see

$$G_t'(\lambda) = ze^{-\xi_t}\exp\Big\{-zh_{0,t}(\lambda)\Big\}h_{0,t}'(\lambda) = z\exp\Big\{-zh_{0,t}(\lambda)\Big\}\frac{\mathrm{d}}{\mathrm{d}u}v_t(0,u,\xi)\bigg|_{u=\lambda e^{\xi_t}},$$

where in the last equality we recall that $h_{0,t}(\lambda) = e^{-\xi_0}v_t(0, \lambda e^{\xi_t}, \xi)$. Moreover, by differentiating with respect to λ on both sides of the backward differential equation (3.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}v_t(0,\lambda,\xi) = 1 - \int_0^t \psi_0'\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)\frac{\mathrm{d}}{\mathrm{d}\lambda}v_t(s,\lambda,\xi)\mathrm{d}s.$$

Thus solving the above equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}v_t(0,\lambda,\xi) = \exp\left\{-\int_0^t \psi_0'\Big(e^{-\xi_s}v_t(s,\lambda,\xi)\Big)\mathrm{d}s\right\}.$$

Then, it follows that

$$\mathbb{E}^{(e,1)}\Big[|G'_t(\lambda)|\Big] = \mathbb{E}^{(e,1)}\Big[G'_t(\lambda)\Big] = z\mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{0,t}(\lambda) - \int_0^t \psi'_0\Big(h_{s,t}(\lambda)\Big)\mathrm{d}s\right\}\right].$$

Observe that $G'_t(\lambda)$ is the Laplace transform given the environment $(\xi_t, t \geqslant 0)$ of a continuous-state branching processes with immigration in random environment, see for instance [38, Theorem 5.3].

In order to find the Riemman integrable function $g(\lambda)$ which dominates the sequence $(\mathbb{E}^{(e,1)}[|G'_t(\lambda)|], t \geqslant 1)$ we use another useful characterisation of $\mathbb{E}^{(e,1)}[G'_t(\lambda)]$. Recalling that the homogeneous Lévy process Ξ (see (5.1)) allows to extend the definition of the mapping $s \mapsto h_{s,0}(\lambda)$ for $s \leqslant 0$, which is the unique positive pathwise solution to (5.3), we write for $\lambda > 0$ and $t \geqslant 0$

$$\mathbb{E}^{(e,1)}\Big[G'_t(\lambda)\Big] = z\mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{-t,0}(\lambda) - \int_{-t}^0 \psi'_0\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right],$$

see Theorem 5.4 and equation (5.6) in [38]. Now, we introduce the function

$$g(\lambda) = \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi'_0\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right].$$

Using the latter characterisation of $\mathbb{E}^{(e,1)}[G'_t(\lambda)]$ together with the non-negative property of ψ'_0 and $h_{-t,0}(\lambda)$, we deduce the following inequality for $t \geqslant 1$

$$\mathbb{E}^{(e,1)}\Big[|G'_t(\lambda)|\Big] \leqslant z\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-t}^0 \psi'_0\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right] \leqslant zg(\lambda).$$

Furthermore, observe that by the assumption in (5.1.1), the function $g(\cdot)$ is Riemann integrable. Next appealing to the Dominated Convergence Theorem in (5.5), we get

$$\begin{aligned}\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) &= \lim_{t\to\infty}\int_0^\infty \mathbb{E}^{(e,1)}\Big[G'_t(\lambda)\Big]\mathrm{d}\lambda \\ &= \int_0^\infty \lim_{t\to\infty}\mathbb{E}^{(e,1)}\Big[G'_t(\lambda)\Big]\mathrm{d}\lambda = \int_0^\infty \mathbb{E}^{(e,1)}\left[\lim_{t\to\infty} G'_t(\lambda)\right]\mathrm{d}\lambda,\end{aligned}$$

where we have used again Dominated Convergence in the last equality since the inequality $|G'_t(\lambda)| \leqslant z$ holds for all $t \geqslant 1$.

On the other hand, also note that assumption $\Phi'_\xi(1) < 0$ implies $\xi_t \to -\infty$ as $t \to \infty$, $\mathbb{P}^{(e,1)}$-a.s. It turns out that $\Xi_t \to \infty$ as $t \to -\infty$, $\mathbb{P}^{(e,1)}$-a.s. Next, thanks to the monotonicity property of the mapping $-t \mapsto v_0(-t, \lambda, \Xi)$ we have

$$h_{-t,0}(\lambda) = e^{-\Xi_{-t}}v_0(-t,\lambda,\Xi) \leqslant e^{-\Xi_{-t}}v_0(0,\lambda,\Xi) = e^{-\Xi_{-t}}\lambda.$$

It follows that $\lim_{t\to\infty} h_{-t,0}(\lambda) = 0$, $\mathbb{P}^{(e,1)}$-a.s., and thus

$$\begin{aligned}\mathbb{E}^{(e,1)}\left[\lim_{t\to\infty} G_t'(\lambda)\right] &= z\mathbb{E}^{(e,1)}\left[\lim_{t\to\infty}\exp\left\{-zh_{-t,0}(\lambda) - \int_{-t}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\\ &= z\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right].\end{aligned}$$

The proof is completed if we show that

$$0 < \mathfrak{B}_2 := \int_0^\infty \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty.$$

From Corollary 5.7 in He et al. [38] (see also the proof of [38, Theorem 5.6]), we see that under moment condition (**H2***), we have

$$\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right] > 0.$$

Therefore

$$\begin{aligned}\int_0^\infty \mathbb{E}^{(e,1)}\left[\lim_{t\to\infty} G_t'(\lambda)\right]\mathrm{d}\lambda &= z\int_0^\infty \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda\\ &= z\mathfrak{B}_2 > 0.\end{aligned}$$

Finally, from the non-negative property of ψ_0' and under the condition in (5.1.1), we obtain the finiteness of $\mathfrak{B}_2$, i.e.,

$$\mathfrak{B}_2 \leqslant \int_0^\infty \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^{0}\psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda = \int_0^\infty g(\lambda)\mathrm{d}\lambda < \infty.$$

This completes the proof.

□

Let us finish this section with the discussion after Theorem 5.1.1 in Section 5.1, concerning the constant $\mathfrak{B}_2$. As we mentioned there, in the general case, it seems difficult to calculate directly the constant $\mathfrak{B}_2$. However, in the stable case we are able to compute it. To this end, we start by noting in the previous proof that

$$z\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right] = \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[G_t'(\lambda)\right],$$

where

$$G_t'(\lambda) = z\exp\Big\{-zh_{0,t}(\lambda)\Big\}\frac{\mathrm{d}}{\mathrm{d}u}v_t(0,u,\xi)\Big|_{u=\lambda e^{\xi_t}}.$$

We also recall that in the stable case (3.27), the backward differential equation in (3.5) can be solved explicitly. More precisely, we have

$$v_t(s,\lambda,\xi) = \Big(\lambda^{-\beta} + \beta C \mathtt{I}_{s,t}(\beta\xi)\Big)^{-1/\beta}.$$

In other words, we obtain

$$\begin{aligned}
&\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right]\\
&\quad = \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[\exp\{-z v_t(0,\lambda e^{\xi_t},\xi)\}\Big(1+(\lambda e^{\xi_t})^{\beta}\beta C\mathtt{I}_{0,t}(\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right].
\end{aligned}$$

Now appealing to the Duality Lemma (3.10) given in Chapter 3, one sees, on the one hand that

$$e^{\beta\xi_t}\mathtt{I}_{0,t}(\beta\xi) = \int_0^t e^{-\beta(\xi_u-\xi_t)}\mathrm{d}u \overset{(d)}{=} \int_0^t e^{\beta\xi_u}\mathrm{d}u = \mathtt{I}_{0,t}(-\beta\xi),$$

and on the other hand,

$$v_t(s,\lambda e^{\xi_t},\xi) = e^{\xi_t}\Big(\lambda^{-\beta}+\beta C e^{\beta\xi_t}\mathtt{I}_{0,t}(\beta\xi)\Big)^{-1/\beta} \overset{(d)}{=} e^{\xi_t}\Big(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\Big)^{-1/\beta}.$$

Hence

$$\begin{aligned}
&\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right]\\
&\quad = \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[\exp\left\{-ze^{\xi_t}\Big(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\Big)^{-1/\beta}\right\}\Big(1+\lambda^{\beta}\beta C\mathtt{I}_{0,t}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right].
\end{aligned}$$

Furthermore, since $\xi_t\to-\infty$ as $t\to\infty$, $\mathbb{P}^{(e,1)}$-a.s., then $\mathtt{I}_{0,\infty}(-\beta\xi)<\infty$, $\mathbb{P}^{(e,1)}$-a.s. Thus, it follows that

$$\lim_{t\to\infty}\exp\left\{-ze^{\xi_t}\Big(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\Big)^{-1/\beta}\right\} = 0, \qquad \mathbb{P}^{(e,1)}-\text{a.s.}.$$

which yields,

$$\mathbb{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right] = \mathbb{E}^{(e,1)}\left[\Big(1+\beta C\lambda^{\beta}\mathtt{I}_{0,\infty}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right].$$

In other words,

$$\begin{aligned}
\mathfrak{B}_2 &= \int_0^\infty \mathbb{E}^{(e,1)}\left[\Big(1+\beta C\lambda^{\beta}\mathtt{I}_{0,\infty}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right]\mathrm{d}\lambda\\
&= (\beta C)^{-1/\beta}\mathbb{E}^{(e,1)}\left[\left(\int_0^\infty e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right],
\end{aligned}$$

where in the last equality we have first used Fubini's Theorem and then we solve the integral with respect to λ.

5.3 Intermediate subcritical regime

The aim of this section is to show Theorem 5.1.2, namely we study the speed of extinction for CSBPs in an Lévy environment in the intermediate subcritical regime. Throughout this section, we assume that the underlying Lévy process ξ fulfils the conditions $\Phi'_\xi(0) < 0$ and $\Phi'_\xi(1) = 0$. In other words, ξ drifts to $-\infty$ under $\mathbb{P}^{(e)}$ and oscillates under the Esscher transform $\mathbb{P}^{(e,1)}$ defined in (3.20).

Before moving to the proof of Theorem 5.1.2, we recall that, under our assumption we have exponential moments of order $\vartheta^+ > 1$, the probability that the supremum of the Lévy process ξ stays below 0, under $\mathbb{P}^{(e,1)}_x$ for $x < 0$, decays as $t^{-1/2}$ up to a multiplicative constant, i.e.

$$\mathbb{P}^{(e,1)}_x\Big(\overline{\xi}_t < 0\Big) \sim \sqrt{\frac{2}{\pi\Phi''_\xi(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]U^{(1)}(-x)t^{-1/2}, \qquad \text{as} \quad t \to \infty, \tag{5.7}$$

where we recall that $U^{(1)}$ denotes the renewal function under $\mathbb{P}^{(e,1)}$ and $(H_t, t \geqslant 0)$ the ascending ladder process, (see Hirano [39, Lemma 11]). For simplicity we split the proof of Theorem 5.1.2 in two lemmas. The first one tell us, under our general assumptions (**H3**) and (3.18) with $\vartheta^- = 0$ and $\vartheta^+ > 1$, that only paths of Lévy processes with a low supremum contribute to the probability of non-extinction.

Lemma 5.3.1. *Suppose that condition* (3.18) *holds with* $\vartheta^- = 0$ *and* $\vartheta^+ > 1$. *We also assume that condition* (**H3**) *is satisfied. Then for any* $z > 0$, $x < 0$ *and* $0 < \delta < 1$, *we have*

$$\lim_{y\to\infty}\limsup_{t\to\infty} t^{1/2}e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y\Big) = 0.$$

Proof. Let $z > 0$, $x < 0$ and $0 < \delta < 1$. Conditioning on ξ and then using the Esscher transform, we deduce that

$$e^{-t\Phi_\xi(1)}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y\Big) \ = \ \mathbb{E}^{(e,1)}_x\Big[e^{-\xi_t}\mathbb{P}_{(z,x)}\Big(Z_t > 0 \ \Big|\ \xi\Big)\mathbf{1}_{\{\overline{\xi}_{t-\delta}\geqslant y\}}\Big].$$

Note that, under the assumption (**H3**), the survival probability conditioned on the environment is bounded from above by the exponential functional of ξ, i.e.,

$$\begin{aligned}\mathbb{P}_{(z,x)}\Big(Z_t > 0 \ \Big|\ \xi\Big) &= 1 - \exp\Big\{ - zv_t(0,\infty,\xi-\xi_0)\Big\} \leqslant zv_t(0,\infty,\xi-\xi_0)\\ &\leqslant z(\beta C)^{-1/\beta}\mathtt{I}_{0,t}(\beta(\xi-\xi_0))^{-1/\beta}.\end{aligned}$$

Similarly as in the previous regime, we appeal to the Duality Lemma given in (3.10) in Chapter 3 and see

$$e^{-\xi_t}\mathtt{I}_{0,t}(\beta\xi)^{-1/\beta} \overset{(d)}{=} \left(\int_0^t e^{\beta\xi_s}\mathrm{d}s\right)^{-1/\beta} = \mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}.$$

The latter implies that

$$\begin{aligned} e^{-t\Phi_\xi(1)}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y\Big) &\leqslant z(\beta C)^{-1/\beta}\mathbb{E}_x^{(e,1)}\Big[e^{-\xi_t}\mathtt{I}_{0,t}(\beta(\xi-\xi_0))^{-1/\beta}\mathbf{1}_{\{\overline{\xi}_{t-\delta}\geqslant y\}}\Big] \\ &= z(\beta C)^{-1/\beta}\mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\mathbf{1}_{\{\underline{\xi}_{t-\delta}\leqslant -y-x\}}\Big]. \end{aligned}$$

According to Li and Xu [52, Lemma 3.5], we have

$$\lim_{y\to\infty}\limsup_{t\to\infty} t^{1/2}\mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\mathbf{1}_{\{\underline{\xi}_{t-\delta}\leqslant -y-x\}}\Big] = 0.$$

Therefore,

$$\begin{aligned} &\lim_{y\to\infty}\limsup_{t\to\infty} t^{1/2}e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y\Big) \\ &\qquad \leqslant z(\beta C)^{-1/\beta}\lim_{y\to\infty}\limsup_{t\to\infty} t^{1/2}\mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}\mathbf{1}_{\{\underline{\xi}_{t-\delta}\leqslant -y-x\}}\Big] = 0, \end{aligned}$$

which concludes the proof. □

Lemma 5.3.2. *Suppose that condition* (**H2***) *holds. We also assume, for $x<0$ that*

$$\int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty. \tag{5.8}$$

Then for every $z>0$ and $x<0$, we have

$$\lim_{t\to\infty} t^{1/2}e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t>0,\ \overline{\xi}_t<0\Big) = z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(x),$$

where

$$\mathfrak{b}_3(x) = U^{(1)}(-x)\mathbb{E}_{-x}^{(e,1),\uparrow}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathrm{d}\lambda\right] \in (0,\infty). \tag{5.9}$$

Proof. Let $z>0$ and assume that $\xi_0 = x < 0$. We begin by recalling that, under

$\mathbb{P}^{(e,1)}$, the Lévy process ξ oscillates. In addition, we have the following identity

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,0)}\Big(Z_t > 0,\ \overline{\xi}_t < -x\Big) \\ &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \ \Big|\ \xi\Big)\mathbf{1}_{\{\overline{\xi}_t < -x\}}\Big] \end{aligned}$$

Recall from (3.8) in Chapter 3, that for any $\lambda \geqslant 0$ and $s \leqslant t$ the cumulant random semigroup $h_{s,t}(\lambda) = e^{-\xi_s}v_t(s, \lambda e^{\xi_t}, \xi)$ satisfies

$$\begin{aligned} \mathbb{E}_{(z,x)}\Big[e^{-\lambda Z_t}\ \Big|\ \xi, \mathcal{F}_s^{(b)}\Big] &= \mathbb{E}_{(z,0)}\Big[e^{-\lambda Z_t e^{\xi_t}e^{-\xi_t}}\ \Big|\ \xi, \mathcal{F}_s^{(b)}\Big] \\ &= \exp\{-Z_s h_{s,t}(\lambda)\}. \end{aligned}$$

It is important to note here that, the initial condition of the Lévy process ξ is irrelevant for the functional $h_{s,t}(\lambda)$. Further, observe that the quenched survival probability of the process $(Z_t, t \geqslant 0)$ is given by $\mathbb{P}_{(z,0)}\Big(Z_t > 0 \ \Big|\ \xi\Big) = 1 - \exp\{-zh_{0,t}(\infty)\}$. Thus,

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \ \Big|\ \xi\Big)\mathbf{1}_{\{\overline{\xi}_t < -x\}}\Big] \\ &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\Big(1 - \exp\Big\{-zh_{0,t}(\infty)\Big\}\Big)\mathbf{1}_{\{\overline{\xi}_t < -x\}}\Big]. \end{aligned}$$

Now, we use the same notation as in the proof of Theorem 5.1.1. Namely, we denote for each fixed $t \geqslant 0$, the function

$$G_t(\lambda) = e^{-\xi_t}\Big(1 - \exp\Big\{-zh_{0,t}(\lambda)\Big\}\Big), \qquad \text{for} \quad \lambda \geqslant 0.$$

Then,

$$G_t(0) = 0 \qquad \text{and} \qquad G_t(\infty) = e^{-\xi_t}\mathbb{P}_{(z,0)}\Big(Z_t > 0 \mid \xi\Big).$$

Since the mapping $\lambda \mapsto h_{0,t}(\lambda)$ is differentiable, then so does $G_t(\cdot)$. In view of the above arguments, we deduce

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= \mathbb{E}^{(e,1)}\Big[\int_0^\infty G_t'(\lambda)\mathrm{d}\lambda \mathbf{1}_{\{\overline{\xi}_t < -x\}}\Big] \\ &= \int_0^\infty \mathbb{E}^{(e,1)}\Big[G_t'(\lambda)\mathbf{1}_{\{\overline{\xi}_t < -x\}}\Big]\mathrm{d}\lambda, \end{aligned}$$

where in the last equality, the expectation and the integral may be exchanged using Fubini's Theorem. Recall the definition of the homogeneous Lévy process Ξ given in (5.1). Now, using the same strategy as in the proof of Theorem 5.1.1, that is, extending the map $s \mapsto h_{s,0}(\lambda)$ for $s \leqslant 0$ and taking the derivate of $G_t(\cdot)$, we have

$$\begin{aligned}\mathbb{E}^{(e,1)}\Big[G_t'(\lambda)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big] &= z\mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{0,t}(\lambda)-\int_0^t \psi_0'\Big(h_{s,t}(\lambda)\Big)\mathrm{d}s\right\}\mathbf{1}_{\{\overline{\xi}_t<-x\}}\right]\\ &= z\mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{-t,0}(\lambda)-\int_{-t}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathbf{1}_{\{\Xi_{-t}>x\}}\right].\end{aligned}$$

To simplify the notation, let us introduce, for $t \geqslant 0$

$$F_t(\lambda) = \exp\left\{-zh_{-t,0}(\lambda)-\int_{-t}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}.$$

Hence, making use of the above observations, we deduce that

$$e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t>0,\ \overline{\xi}_t<0\Big) = z\mathbb{P}^{(e,1)}\Big(\Xi_{-t}>x\Big)\int_0^\infty \mathbb{E}^{(e,1)}\Big[F_t(\lambda)\ \Big|\ \Xi_{-t}>x\Big]\mathrm{d}\lambda.$$

Now, taking into account that

$$\lim_{t\to\infty} t^{1/2}\mathbb{P}^{(e,1)}\Big(\Xi_{-t}>x\Big) = \lim_{t\to\infty} t^{1/2}\mathbb{P}^{(e,1)}\left(\overline{\xi}_t<-x\right) = \sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]U^{(1)}(-x), \tag{5.10}$$

thus the proof of this lemma will be completed once we have shown

$$\begin{aligned}\lim_{t\to\infty}\int_0^\infty \mathbb{E}^{(e,1)}\Big[F_t(\lambda)\ \Big|\ \Xi_{-t}>x\Big]\mathrm{d}\lambda &= \lim_{t\to\infty}\int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\ \Big|\ \Xi_{-t}>0\Big]\mathrm{d}\lambda\\ &= \mathbb{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathrm{d}\lambda\right]\\ &=: b(x).\end{aligned}$$

The arguments used to deduce the preceding limit are quite involved, for that reason we present its proof in three steps.

Step 1. Let us first introduce the following functions, for $r, \lambda \geqslant 0$ and $t \geqslant 0$,

$$f_r(t,\lambda) = \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\ \Big|\ \Xi_{-(t+r)}>0\Big],$$

$$g_r(t,\lambda) = \mathbb{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-t}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\ \middle|\ \Xi_{-(t+r)}>0\right].$$

Since $F_t(\lambda)$ and $\exp\{-\int_{-t}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}\lambda\}$ are bounded random variables, we can proceed similarly as in Lemma 4.3.1, to deduce that as $r \to \infty$

$$f_r(t,\lambda) \to \mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_t(\lambda)\right] \quad \text{and} \quad g_r(t,\lambda) \to \mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-t}^{0} \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right],$$

as well as the following upper bound

$$g_r(t,\lambda) \leqslant C_1(t)\mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-t}^{0} \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right],$$

where $C_1(t)$ is a positive constant which depends on t, see inequality (4.23). We may now appeal to Dominated Convergence Theorem together with our hypothesis (5.8), to deduce that for $t \geqslant 1$

$$\int_0^\infty g_r(t,\lambda)\mathrm{d}\lambda \to \int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-t}^{0} \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda, \quad \text{as} \quad r \to \infty.$$

Furthermore, since $f_r(t,\lambda) \leqslant g_r(t,\lambda)$, an application of the *generalised* Dominated Convergence Theorem (see for instance Folland [30, Exercise 2.20]) tell us

$$\lim_{r\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\left[F_t(\lambda) \;\Big|\; \Xi_{-(t+r)} > 0\right]\mathrm{d}\lambda = \int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_t(\lambda)\right]\mathrm{d}\lambda. \tag{5.11}$$

Step 2. Let $1 \leqslant s \leqslant t$, $\lambda \geqslant 0$ and $\gamma \in (1,2]$. From the proof of [7, Lemma 4], we can deduce

$$\left|\mathbb{E}_{-x}^{(e,1)}\left[F_t(\lambda) - F_s(\lambda) \;\Big|\; \Xi_{-\gamma t} > 0\right]\right| \leqslant C_2\mathbb{E}_{-x}^{(e,1),\uparrow}\Big[|F_t(\lambda) - F_s(\lambda)|\Big],$$

where C_2 is a positive constant. Hence

$$\left|\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\left[F_t(\lambda) - F_s(\lambda) \;\Big|\; \Xi_{-\gamma t} > 0\right]\mathrm{d}\lambda\right| \leqslant C_2\int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow}\Big[|F_t(\lambda) - F_s(\lambda)|\Big]\mathrm{d}\lambda.$$

Now, under the event that $\{\Xi_{-\gamma t} > 0\}$, we know that, for each $\lambda \geq 0$, the inequalities $h_{-s,0}(\lambda) \leqslant \lambda e^{-\Xi_{-s}} \leqslant \lambda$ hold. It implies that under $\{\Xi_{-\gamma t} > 0\}$, we obtain

$$\begin{aligned}
|F_t(\lambda) - F_s(\lambda)| &= \exp\left\{-\int_{-s}^{0} \psi_0'(h_{u,0}(\lambda))\mathrm{d}u\right\} \\
&\qquad \left|\exp\left\{-zh_{-t,0}(\lambda) - \int_{-t}^{-s} \psi_0'\Big(h_{u,0}(\lambda)\Big)\mathrm{d}u\right\} - \exp\{-zh_{-s,0}(\lambda)\}\right| \\
&\leqslant \exp\left\{-\int_{-s}^{0} \psi_0'(h_{u,0}(\lambda))\mathrm{d}u\right\}\left|\exp\left\{-\int_{-t}^{-s} \psi_0'\Big(h_{u,0}(\lambda)\Big)\mathrm{d}u\right\} - e^{-z\lambda}\right| \\
&\leqslant 2\exp\left\{-\int_{-1}^{0} \psi_0'(h_{u,0}(\lambda))\mathrm{d}u\right\}.
\end{aligned}$$

It then follows, from the previous calculations and our assumption (5.8) together with Dominated Convergence Theorem, that

$$\lim_{s\to\infty}\lim_{t\to\infty}\left|\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)-F_s(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda\right|=0,$$

which in particular yields

$$\lim_{s\to\infty}\lim_{t\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)-F_s(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda=0.$$

Thus, appealing to (5.11) in Step 1, we get

$$\begin{aligned}\lim_{t\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda &= \lim_{s\to\infty}\lim_{t\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\Big[F_s(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda\\ &= \lim_{s\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_s(\lambda)\right]\mathrm{d}\lambda.\end{aligned}$$

In order to deal with the limit in the right-hand side, first note that $h_{-s,0}(\lambda)\leqslant\lambda e^{-\Xi_{-s}}\to 0$ as $s\to\infty$, $\mathbb{P}_{-x}^{(e,1)}$-a.s. Moreover, we have

$$\mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_s(\lambda)\right]\to\mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-\infty}^0\psi_0'(h_{u,0}(\lambda))\mathrm{d}\lambda\right\}\right]\quad\text{as}\quad s\to\infty,$$

and for $s\geqslant 1$

$$\mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_s(\lambda)\right]\leqslant\mathbb{E}_{-x}^{(e,1),\uparrow}\left[\exp\left\{-\int_{-1}^0\psi_0'(h_{u,0}(\lambda))\mathrm{d}\lambda\right\}\right].$$

Hence, we may now apply once again the Dominated Convergence Theorem to deduce that

$$\int_0^\infty\lim_{s\to\infty}\mathbb{E}_{-x}^{(e,1),\uparrow}\left[F_s(\lambda)\right]\mathrm{d}\lambda=b(x)<\infty.$$

In other words, we have

$$\lim_{t\to\infty}\int_0^\infty \mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda=b(x).$$

Next, from (5.10) we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\mathbb{P}_{-x}^{(e,1)}(\Xi_{-t}>0)}\int_0^\infty\mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda\\ &\qquad=\lim_{t\to\infty}\frac{\mathbb{P}_{-x}^{(e,1)}\big(\Xi_{-\gamma t}>0\big)}{\mathbb{P}_{-x}^{(e,1)}\big(\Xi_{-t}>0\big)}\int_0^\infty\mathbb{E}_{-x}^{(e,1)}\Big[F_t(\lambda)\;\Big|\;\Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda\\ &\qquad=\gamma^{-\rho}b(x).\end{aligned}$$

Since γ may be chosen arbitrarily close to 1, we have

$$\int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda - b(x)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big) = o(1)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big).$$

Step 3. Let $\lambda \geqslant 0$, $t \geqslant 1$ and $\gamma \in (1,2]$ and denote

$$J_t(\lambda) = \frac{1}{\mathbb{P}^{(e,1)}_{-x}(\Xi_{-t}>0)}\mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\left(\mathbf{1}_{\{\Xi_{-t}>0\}} - \mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\right)\Big].$$

Note that, from (5.10) and since $F_t(\lambda) \leqslant 1$, we get

$$0 \leqslant J_t(\lambda) \leqslant 1 - \frac{\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-\gamma t}>0\Big)}{\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big)} \to 1-\gamma^\rho, \quad \text{as} \quad t\to\infty.$$

Since γ may be taken arbitrary close to 1, we deduce that $J_t(\lambda) \to 0$ as $t\to\infty$. In addition,

$$\begin{aligned} J_t(\lambda) &\leqslant \mathbb{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-t}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\} \,\Big|\, \Xi_{-t}>0\right] \\ &\leqslant \mathbb{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\} \,\Big|\, \Xi_{-t}>0\right] \\ &\leqslant C_3\mathbb{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h_{s,0}(\lambda))\mathrm{d}s\right\}\right], \end{aligned}$$

where C_3 is a positive constant and the right-hand side is an integrable function in λ thanks to the assumption (5.8). Hence, appealing to the Dominate Convergence Theorem, we see

$$\int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\left(\mathbf{1}_{\{\Xi_{-t}>0\}} - \mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\right)\Big]\mathrm{d}\lambda = o(1)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big).$$

We combine the previous limit with the conclusion of Steps 2 to conclude, as promised earlier, that

$$\begin{aligned} &\int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-t}>0\}}\Big]\mathrm{d}\lambda - b(x)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big) \\ &\qquad = \int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-t}>0\}}\Big]\mathrm{d}\lambda - \int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda \\ &\qquad\qquad + \int_0^\infty \mathbb{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda - b(x)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big) \\ &\qquad = o(1)\mathbb{P}^{(e,1)}_{-x}\Big(\Xi_{-t}>0\Big). \end{aligned}$$

Finally, similarly as in the proof of Theorem 5.1.1, we see that the moment condition (**H2***) guarantees that $b(x) > 0$. This concludes the proof. $\square$

Proof of Theorem 5.1.2. The proof of this result essentially mimics the steps of Theorem 5.1.1. More precisely, let $z, \epsilon > 0$ and $x < 0$. From Lemma 5.3.1, we have for every $\delta \in (0,1)$,

$$\lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} e^{-t\Phi_\xi(\gamma)} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y \Big) = 0.$$

Then it follows that, we may choose $y > 0$ such that for t sufficiently large

$$\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y \Big) \leqslant \epsilon \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} < y \Big).$$

Further, since $\{Z_t > 0\} \subset \{Z_{t-\delta} > 0\}$ for t large, we deduce that

$$\begin{aligned} \mathbb{P}_z(Z_t > 0) &= \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geqslant y \Big) + \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} < y \Big) \\ &\leqslant (1+\epsilon)\mathbb{P}_{(z,x-y)}\Big(Z_{t-\delta} > 0,\ \overline{\xi}_{t-\delta} < 0 \Big). \end{aligned}$$

In other words, for every $\epsilon > 0$ there exists $y' < 0$ such that

$$\begin{aligned} &(1-\epsilon) t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,y')}\Big(Z_t > 0,\ \overline{\xi}_t < 0 \Big) \leqslant t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_z(Z_t > 0) \\ &\leqslant (1+\epsilon)(t-\delta)^{1/2} e^{-\Phi_\xi(1)(t-\delta)} \mathbb{P}_{(z,y')}\Big(Z_{t-\delta} > 0,\ \overline{\xi}_{t-\delta} < 0 \Big) \frac{t^{1/2} e^{-\Phi_\xi(1)t}}{(t-\delta)^{1/2} e^{-\Phi_\xi(1)(t-\delta)}}. \end{aligned}$$

Now we use Lemma 5.3.2 and the same reasoning as in the last steps in the proof of Theorem 5.1.1 to get the desired result. More precisely, appealing to Lemma 5.3.2, we have

$$\lim_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,y')}\Big(Z_t > 0,\ \overline{\xi}_t < 0 \Big) = z\sqrt{\frac{2}{\pi \Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\Big[H_1 \Big] \mathfrak{b}_3(y'),$$

where

$$\mathfrak{b}_3(y') = U^{(1)}(-y') \mathbb{E}^{(e,1),\uparrow}_{-y'} \left[\int_0^\infty \exp\left\{ -\int_{-\infty}^0 \psi_0'\Big(h_{s,0}(\lambda) \Big) \mathrm{d}s \right\} \mathrm{d}\lambda \right]. \tag{5.12}$$

Hence, we obtain

$$(1-\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y')\leqslant \lim_{t\to\infty} t^{1/2}e^{-t\Phi_\xi(1)}\mathbb{P}_z(Z_t>0)$$
$$\leqslant (1+\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y')e^{-\Phi_\xi(1)\delta}.$$

On the other hand, we observe that y' is a sequence which may depend on ϵ . Further, this sequence y' goes to minus infinity as ϵ goes to 0. Then, for any sequence $y'=y_\epsilon$, we have

$$0<(1-\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y_\epsilon)\leqslant \lim_{t\to\infty} t^{3/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t>0)$$
$$\leqslant (1+\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y_\epsilon)<\infty.$$

Therefore, by letting $\epsilon\to 0$, we get

$$0<\liminf_{\epsilon\to 0}(1-\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y_\epsilon)\leqslant \lim_{t\to\infty} t^{3/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t>0)$$
$$\leqslant \limsup_{\epsilon\to 0}(1+\epsilon)z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{b}_3(y_\epsilon)e^{-\Phi_\xi(1)\delta}<\infty.$$

Since δ can be taken arbitrary close to 0, we deduce

$$\lim_{t\to\infty} t^{3/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t>0)=z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\Big[H_1\Big]\mathfrak{B}_3,$$

where

$$\mathfrak{B}_3:=\lim_{\epsilon\to 0}\mathfrak{b}_3(y_\epsilon)=\lim_{\epsilon\to 0}U^{(1)}(-y_\epsilon)\mathbb{E}^{(e,1),\uparrow}_{-y_\epsilon}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\Big(h_{s,0}(\lambda)\Big)\mathrm{d}s\right\}\mathrm{d}\lambda\right].$$

Thus the proof is completed.

□

5.4 Conjecture: weakly subcritical regime

In order to prove Conjecture 5.1.1, we believe that it is possible to adapt our arguments developed in the intermediate-subcritical regime for the non-extinction probability and combine with the approach introduced in [1], for the discrete time setting.

Roughly speaking, the aim is to study the event of non-extinction at time t in two different situations that depend on the behaviour of the infimum of the environment. To be more precise, we split the event of non-extinction as follows:

$$\mathbb{P}_z(Z_t > 0) = \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \underline{\xi}_t > 0\Big) + \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \underline{\xi}_t \leqslant 0\Big), \tag{5.13}$$

for $z, x > 0$. For the asymptotic behaviour of the second probability in the right-hand side above, it will be necessary to use condition (**H3**) which give us the following lower bound

$$v_t(0, 0, \xi - \xi_0) \leqslant \Big(C\beta \mathtt{I}_{0,t}(\beta(\xi - \xi_0))\Big)^{-1/\beta}.$$

In addition, from Lemma 4.3 in [52], we have that there exists t_0 and a constant $C_\beta > 0$ such that for $t > t_0$,

$$t^{3/2} e^{-\Phi_\xi(\gamma)t} \mathbb{E}^{(e)} \left[\mathtt{I}_{0,t}(\beta\xi)^{-1/\beta}\right] \leq C_\beta.$$

With these two facts in hand, it will be possible to handle the expectation

$$\mathbb{E}_x^{(e)}\Big[\mathbb{P}_{(z,x)}\Big(Z_t < \infty \ \Big|\ \xi\Big)\mathbf{1}_{\{\bar{\xi}_t \geqslant 0\}}\Big],$$

in order to obtain its long-term behaviour.

Now, for the first probability in the right-hand side of (5.13), a more elaborate argument will be needed. To this end, we write

$$\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \underline{\xi}_t > 0\Big) = \mathbb{P}_{(z,x)}\Big(Z_t > 0 \ \Big|\ \underline{\xi}_t > 0\Big)\mathbb{P}_x^{(e)}\Big(\underline{\xi}_t > 0\Big).$$

We know that the probability that the running infimum of the environment remains positive decays exponentially with a factor of order $t^{-3/2}$ up to a multiplicative constant. More precisely, for $x > 0$

$$\mathbb{P}_x^{(e)}\left(\underline{\xi}_t > 0\right) \sim A_\gamma e^{\gamma x} \widehat{U}^{(\gamma)}(x) t^{-3/2} e^{\Phi_\xi(\gamma)t} \int_0^\infty e^{-\gamma z} U^{(\gamma)}(z)\mathrm{d}z, \quad \text{as} \qquad t \to \infty,$$

where $\widehat{U}^{(\gamma)}$ is the renewal function defined in (3.20) with $\theta = \gamma$ in the Esscher transform and

$$A_\gamma := \frac{1}{\sqrt{2\pi\Phi''_\xi(\gamma)}} \exp\left\{\int_0^\infty (e^{-t}-1)t^{-1}e^{-t\Phi_\xi(\gamma)}\mathbb{P}^{(e)}(\xi_t = 0)\mathrm{d}t\right\}.$$

The latter asymptotic behaviour should be the one leading the asymptotic behaviour of the probability of non-extinction in this particular case. Therefore, for the conditional probability

$$\mathbb{P}_{(z,x)}\Big(Z_t > 0 \;\Big|\; \underline{\xi}_t > 0\Big),$$

it will be convenient to show that it converges to a positive constant as t increases. In order to do so, it will be necessary to study the CSBP in an environment conditioned to be positive. To this end, we will appeal to the fluctuation theory of Lévy processes together with the ideas developed by Afanasyev et al. in [1], for the discrete time setting.

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