A Note on Characterizing Tightness of Random Sets of Càdlàg Paths

Nic Freeman

School of Mathematics and Statistics, University of Sheffield, UK
n.p.freeman@sheffield.ac.uk

Sandra Palau

Department of Statistics and Probability
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México, México
sandra@sigma.iimas.unam.mx

We record two natural tightness criteria for random sets of càdlàg paths, within the state space introduced by Etheridge et al. in [4]. This state space is the equivalent, for càdlàg paths, of the (continuous path) state space introduced for the Brownian web by Fontes et al. in [6].

We discuss some implications of these criteria, in particular for cases of coalescing random walkers with heavy tailed jumps, in which tightness fails.

1. Introduction

In this article, we focus on an extension of the "usual" state space for the Brownian web that was introduced by [6]; the extension, which was introduced by [4], allows for the possibility that paths may be càdlàg, but otherwise preserves direct analogy to the usual state space. We record two natural characterizations of tightness for random sets of càdlàg paths, in the style of the classical characterizations of tightness for (single) random càdlàg paths. Using these characterizations, we discuss a corresponding failure of tightness for coalescing random walk approximations to webs made up of α-stable Lévy process, which were introduced recently by [9].

This article developed out of discussions following a learning session, given by the present authors, on the Brownian web and net, at the Workshop on Genealogies of Interacting Particle Systems, held at the Institute for Mathematic Sciences in Singapore. The material used in our learning session, which provides a brief introduction to the theory of the Brownian web and net, are available online as [7]. The reader is referred to [11] for a comprehensive introduction to the Brownian web and related objects. However, in this article, we are interested only in their state spaces.

Let us begin by setting up key pieces of notation and briefly describing the state space introduced by [6]. For $s \in [-\infty, \infty]$, let

$$C[s] = \big\{ f : [s, \infty] \to [-\infty, \infty] \,;\, f \text{ is continuous on } [s, \infty] \cap (-\infty, \infty) \big\}$$
$$D[s] = \big\{ f : [s, \infty] \to [-\infty, \infty] \,;\, f \text{ is càdlàg on } [s, \infty] \cap (-\infty, \infty) \big\}.$$

For $f \in D[s]$ we write $\sigma_f = s$ for the first time at which f is defined. We set

$$\widetilde{\Pi} = \bigcup_{t \in [-\infty, \infty]} C[t], \qquad \Pi = \bigcup_{t \in [-\infty, \infty]} D[t].$$

For each $f \in D[s]$ we associate f to a function $\bar{f} : [\sigma_{\bar{f}}, 1] \to [-1, 1]$, by applying the space-time transformation

$$(x, t) \mapsto \left(\frac{\tanh(x)}{1 + |t|}, \tanh(t) \right) \tag{1.1}$$

which maps the space-time plane from $[-\infty, \infty]^2$ into a subset of $[-1, 1]^2$ (see Figure 1). The choice of the function tanh is arbitrary: it could be replaced by any order preserving continuous bijection of $[-\infty, \infty]$ to $[-1, 1]$.

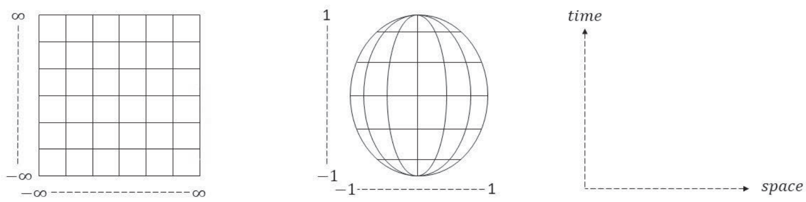

Fig. 1. The space-time transformation (1.1): before (left), after (mid), axis labels (right).

Formally, for each $s \in [-\infty, \infty]$ and $f \in D[s]$ we define $\bar{f}$ as follows. Let $\kappa_t = \tanh^{-1}(t)$ and note that κ is an order preserving homeomorphism between $[-1, 1]$ and $[-\infty, \infty]$ (we use the symbol κ in place of $\tanh^{-1}$ to denote a change of time rather than rescaling of space). Then for $f \in \Pi$ we define

$$\bar{f}(t) = \frac{\tanh(f(\kappa_t))}{1 + |\kappa_t|} \tag{1.2}$$

for $t \in [\kappa^{-1}(\sigma_f), 1]$, with the convention $1/\infty = 0$. It follows immediately that $\bar{f}$ is càdlàg. Moreover, if $f \in \widetilde{\Pi}$, then $\bar{f}$ is continuous.

We use the map $f \mapsto \bar{f}$ to induce a pseudo-metric on $\widetilde{\Pi}$

$$d_{\widetilde{\Pi}}(f_1, f_2) = |\sigma_{\bar{f}_1} - \sigma_{\bar{f}_2}| \vee \sup_{t \in [-1,1]} \left| \bar{f}_1\left(t \vee \sigma_{\bar{f}_1}\right) - \bar{f}_2\left(t \vee \sigma_{\bar{f}_2}\right) \right|. \tag{1.3}$$

In standard fashion, we associate each $f \in \widetilde{\Pi}$ with its corresponding equivalence class and, with slight abuse of notation, regard $d_{\widetilde{\Pi}}$ as a metric on $\widetilde{\Pi}$. Essentially, $d_{\widetilde{\Pi}}$ is the usual supremum metric on continuous paths, ignoring behaviour near $\pm\infty$ in both space and time, and modified to handle paths with possibly different starting times.

The metric $d_{\widetilde{\Pi}}$ is that of [6], although their notation is a little different. It was shown in [6] that $(\widetilde{\Pi}, d_{\widetilde{\Pi}})$ is both complete and separable. The space $\mathcal{K}(\widetilde{\Pi})$ is defined to be the space of compact subsets of $\widetilde{\Pi}$, equipped with the induced Hausdorff metric, and including the empty set as an isolated point.

Thus, an element of $\mathcal{K}(\widetilde{\Pi})$ is a (compact) set of continuous paths. An analogous construction for càdlàg paths was introduced by [4]; it replaces the sup in (1.3) with a function based on the Skorohod metric, resulting in a space $\mathcal{K}(\Pi)$ whose elements are (suitably compact) sets of càdlàg paths. We will recall the detail of this construction in Section 1.1.

We now look to state our pair of characterizations of tightness in $\mathcal{K}(\Pi)$. In order to explain where they come from, we will first recall tightness characterizations for $\mathcal{K}(\widetilde{\Pi})$ and for spaces of (single) continuous/càdlàg paths. The reader who is already familiar with such results may wish to simply skip forwards to equation (1.12).

Let $\mathscr{C}$ denote the space of continuous functions $f : [-1, 1] \to [-1, 1]$, with the usual supremum metric. Theorem 7.3 of [3] provides a characterization of tightness in $\mathscr{C}$: a random sequence (X_n) with values in $\mathscr{C}$ is tight if and only if, for all $\eta > 0$,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left[w(X_n, \delta) > \eta\right] = 0, \tag{1.4}$$

where

$$w(f, \delta) = \sup_{|s-t|<\delta} |f(s) - f(t)| \tag{1.5}$$

is the modulus of continuity of f. The supremum in (1.5) ranges over all s, t in the domain of f for which $|s-t| < \delta$: we adopt the convention that whenever an expression of the form $\sup_{\ldots}(\ldots)$ contains $f(\cdot)$, the argument of f is automatically restricted to its domain, and similarly for inf.

Let $\Lambda_{L,T} = \{(x,t)\,;\,|x| \le L, |t| \le T\}$. Extending the same ideas to $\mathcal{K}(\widetilde{\Pi})$, as in Section 6.1 of [11], it turns out that what is required to have tightness is uniform control of the modulus of continuity: a random sequence $\widetilde{\mathcal{X}}_n$ in $\mathcal{K}(\widetilde{\Pi})$ is tight if and only if, for all $\eta > 0$, and for all bounded $\Lambda_{L,T}$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left[\sup_{f\in\widetilde{\mathcal{X}}_n} w(f,\delta,\Lambda_{L,T}) > \eta\right] = 0. \tag{1.6}$$

Here,

$$w(f,\delta,\Lambda_{L,T}) = \sup_{\substack{|s-t|<\delta \\ (f(t),t),(f(s),s)\in\Lambda_{L,T}}} |f(s)-f(t)|, \tag{1.7}$$

which one might think of (loosely) as a way of restricting $w(f,\delta)$ to only care about behaviour that originates from within the box $\Lambda_{L,T}$.

In another vein, if we let $\mathscr{D}$ denote the set of càdlàg functions $f : [-1,1] \to [-1,1]$, with the usual Skorohod metric, then it is well known that a sequence X_n of random elements of $\mathscr{D}$ is tight if and only if, for all $\eta > 0$

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}\left[w'(X_n,\delta) > \eta\right] = 0 \tag{1.8}$$

where

$$\begin{aligned} w'(f,\delta) &= \inf_{\{t_i\}}\max_{1\le i<m} w_i(f,\delta) \\ w_i(f,\delta) &= \sup_{\substack{|s-t|<\delta \\ s,t\in[t_i,t_{i+1})}} |f(s)-f(t)|. \end{aligned} \tag{1.9}$$

Here, the infimum ranges over all finite sequences $\{t_i\}_{i=1}^m$ such that $-1 = t_1 < t_2 < \cdots < t_m = 1$ and $\min_i |t_{i+1} - t_i| \ge \delta$. We refer to such sequences as δ-sparse partitions (of $[-1,1]$). The function w' plays the role of the modulus of continuity for càdlàg paths. Essentially, w' differs from w by permitting f to make jumps (at times t_i) without these jumps increasing the value of w'.

It is natural to expect that the combination of these two extensions of (1.4) should result in a characterization of tightness for sequences of random sets of càdlàg paths. It does, but:

Remark 1.1: In (1.7), with continuous paths, we did not need to consider the possibility of paths making large jumps between $-\infty$ and ∞ (or points close to $\pm\infty$) without passing through $\Lambda_{L,T}$ on the way. With càdlàg paths, we must account for this possibility.

With the above remark in mind, for $f \in \Pi$, define $f_{L,T}$, with the same domain as f, by

$$f_{L,T}(t) = (-L) \vee f\Big((-T) \vee t \wedge T\Big) \wedge L. \tag{1.10}$$

We refer to $f_{L,T}$ as the restriction of f to $\Lambda_{L,T}$.

It is straightforward to check that $w(f, \delta, \Lambda_{L,T})$ in (1.7) could be replaced by

$$\hat{w}(f, \delta, \Lambda_{L,T}) = \sup_{|s-t|<\delta} |f_{L,T}(s) - f_{L,T}(t)|, \tag{1.11}$$

to characterize tightness in $\mathcal{K}(\widetilde{\Pi})$. It is this formulation which extends to $\mathcal{K}(\Pi)$.

For the space we are primarily interested in, $\mathcal{K}(\Pi)$, the natural equivalent of the modulus of continuity is

$$w'(f, \delta, \Lambda_{L,T}) = \inf_{\{t_i\}} \max_{1\le i<m} w_i(f, \delta, \Lambda_{L,T}) \tag{1.12}$$

$$w_i(f, \delta, \Lambda_{L,T}) = \sup_{\substack{|s-t|<\delta \\ s,t\in[t_i,t_{i+1})}} |f_{L,T}(s) - f_{L,T}(t)|. \tag{1.13}$$

The infimum in (1.12) ranges over δ-sparse partitions of $[\sigma_f, T]$, i.e. $\sigma_f = t_1 < \cdots < t_m = T$ and $\min_i |t_{i+1} - t_i| \ge \delta$. Thus:

Proposition 1.2: *Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be a sequence of random elements of $\mathcal{K}(\Pi)$. Then $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is tight if and only if, for all $\eta > 0$ and all bounded $\Lambda_{L,T}$,*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left[\sup_{f\in\mathcal{X}_n} w'(f, \delta, \Lambda_{L,T}) > \eta\right] = 0. \tag{1.14}$$

A second characterization of tightness in the Skorodhod space $\mathscr{D}$ is often useful, and provides a version of the modulus of continuity that does not rely on δ-sparse partitions. In particular, from Theorem 12.4 of [3], it can be seen that function w' in (1.9) can be replaced by

$$w''(f, \delta) = \sup_{\substack{s_1\le t\le s_2 \\ |s_2-s_1|<\delta}} |f(s_1) - f(t)| \wedge |f(t) - f(s_2)|, \tag{1.15}$$

plus the additional requirements of uniform right continuity at -1 and uniform left limits at 1.

When compared to (1.5), w'' has a natural interpretation: in the world of continuous paths, tightness fails if the sequence X_n contains a jump that does not vanish in the limit; in the world of càdlàg paths tightness fails if, in the limit, the sequence X_n contains two jumps, occurring arbitrarily close together, neither of which vanishes in the limit.

Extending to $\mathcal{K}(\Pi)$, the natural equivalent of (1.15) is

$$w''(f,\delta,\Lambda_{L,T}) = \sup_{\substack{s_1\le t\le s_2\\ |s_2-s_1|<\delta}} \big|f_{L,T}(s_1)-f_{L,T}(t)\big| \wedge \big|f_{L,T}(t)-f_{L,T}(s_2)\big|. \quad (1.16)$$

Additionally, we will need a way to require (uniform) right continuity at the starting time of f,

$$w^+(f,\delta,\Lambda_{L,T}) = \sup_{s\in[0,\delta)} \big|f_{L,T}(\sigma_f) - f_{L,T}(\sigma_f+s)\big|. \quad (1.17)$$

We won't need to consider the end time of f, because (1.1) pinches space to a point at time $+\infty$ and consequently enforces left continuity there.

We obtain:

Proposition 1.3: *Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ be a sequence of random elements of $\mathcal{K}(\Pi)$. Then $(\mathcal{X}_n)_{n\in\mathbb{N}}$ is tight if and only if, for all $\eta>0$ and all bounded $\Lambda_{L,T}$,*

$$\lim_{\delta\to0}\limsup_{n\to\infty}\mathbb{P}\left[\sup_{f\in\mathcal{X}_n}\left(w''(f,\delta,\Lambda_{L,T})\vee w^+(f,\delta,\Lambda_{L,T})\right)>\eta\right]=0. \quad (1.18)$$

Since it does not use δ-sparse partitions, the characterization in Proposition 1.3 may be better suited for use in one type of argument that has often been used to prove tightness for random sets of paths i.e. cutting space-time up into small regions and looking to control macroscopic jumps originating from within each region – see e.g. Section 6.1 of [11] for details in the case of the Brownian web.

Propositions 1.2 and 1.3 are not deep results, but they are nonetheless useful. The proofs are closely connected to the classical theory of tightness in path-space, and we provide them in Section 2.

We note that the state space $\widetilde{\Pi}$ introduced by [6] has become widely used, but it is by no means the only suitable state space for random sets of paths such as the Brownian web. Several authors, including [12] and [10] have used state spaces based on stochastic flows, in which case special care is needed to handle the existence of points within the Brownian web with more than one (equivalence class of) outgoing paths. A second alternative, used in e.g. [8], is the 'marked measure space', which seeks to characterize a set of paths in terms of the times until meetings of paths emanating from finite, random, sets of points within the underlying space.

Recently, [2] proposed a new state space based on the concept of 'tubes'. In their topology, sets of paths are characterized by which (tube-shaped) regions of space they pass through. The resulting state space has many appealing properties, including compactness of the state space itself, which makes tightness automatic.

A further example is provided by a construction of the α-stable web in [9]. Although [9] does introduce a space that is essentially equivalent to our $\mathcal{K}(\Pi)$, using a metric analogous to that defined below in (1.20), their construction of the α-stable web then takes place under a *weaker* topology. Loosely speaking, they go on to introduce an auxiliary metric which, in a sense that we won't describe fully here, simply ignores the positions of paths until after they have coalesced with an older path. In this weakened topology, [9] proves tightness by constructing an explicit family of suitable relatively compact sets, rather than via the sort of general characterisation stated above.

In fact, for heavy tailed random walk approximations to the α-stable web, tightness will fail in $\mathcal{K}(\Pi)$. This failure is closely related to the $3-\epsilon$ moment condition that was discovered by [1] to be necessary for convergence in $\mathcal{K}(\widetilde{\Pi})$ of coalescing random walks to the Brownian web. We demonstrate this failure of tightness in Section 3, in which the discussion relies heavily on applying equation (1.6) and Proposition 1.3, but is otherwise self-contained.

1.1. *A state space for sets of càdlàg paths*

In this section we recall the spaces Π and $\mathcal{K}(\Pi)$ introduced by [4] (in which they were denoted M and $\mathcal{K}(M)$). We begin by considering the space

$$G = \{g : [\sigma_g, 2] \to [-1,1]\,;\, g \text{ is càdlàg},\ \sigma_g \in [-1,1],\ g \text{ is constant on } [1,2]\}\,. \tag{1.19}$$

We wish to view G as a space of càdlàg paths, with possibly different starting times, and we now set about defining a metric which embodies this intuition.

For $g, h \in G$, let $\Lambda'[g,h]$ denote the set of strictly increasing bijections from $[\sigma_g, 2] \to [\sigma_h, 2]$. We define $\Lambda[g,h] \subseteq \Lambda'[g,h]$ to be the subset of

$\lambda \in \Lambda'[g,h]$ for which

$$\gamma_{g,h}(\lambda) = \sup_{\sigma_g \le t < s \le 2} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| < \infty.$$

For such g, h, λ we define

$$d(g,h,\lambda) = \sup_{t \in [\sigma_g, 2]} |g(t) - h(\lambda(t))|,$$

and

$$\rho(g,h) = \inf_{\lambda \in \Lambda[g,h]} \Big(\gamma_{g,h}(\lambda) \vee d(g,h,\lambda) \Big). \tag{1.20}$$

Remark 1.4: If we used the domain $[\sigma_g, 1]$ for $g \in G$ in (1.19), instead of $[\sigma_g, 2]$, $\Lambda'[g,h]$ would be empty whenever $\sigma_h < \sigma_g = 1$. To avoid this technicality, we work over time $[-1, 2]$ and require constancy on $[1, 2]$.

It was shown in Section 5 of [4] that G is complete and separable under the metric

$$d_G(g,h) = \rho(g,h) \vee |\sigma_g - \sigma_h|.$$

Intuitively, this metric says that paths in G converge if their domains converge and, as the domains become close, the paths also become close (in the Skorohod sense). It is the direct equivalent, for càdlàg paths, to the metric (1.3) introduced by [6] for the "usual" state space of continuous paths used for the Brownian web. In fact, if we restrict to paths starting at some fixed time, d_G becomes the usual Skorohod metric. The metric d_G is the key ingredient for metrizing Π.

Recall $\bar{f}$ from (1.2). With slight abuse of notation, we extend each $\bar{f}$ to a function $\bar{f} \in G$ by setting $\bar{f}(t) = 0$ for all $t \in (1, 2]$. Then,

$$d_\Pi(f_1, f_2) = d_G(\bar{f}_1, \bar{f}_2) \tag{1.21}$$

is a pseudo-metric on Π. In standard fashion, from now on we implicitly work with equivalence classes of Π and, with mild abuse of notation, treat (Π, d_Π) as a metric space. Separability of G immediately implies the same for Π, and it was shown in Lemma 5.6 of [4] that completeness is also inherited by Π. Of course, convergence in (Π, d_Π) can be described as local Skorohod convergence of the paths plus convergence of the starting times. It was shown in Lemma 5.8 of [4] that $\widetilde{\Pi}$ is continuously embedded in Π.

Recall that $\mathcal{K}(\Pi)$ is the set of compact subsets of Π, equipped with the Hausdorff metric $d_{\mathcal{K}(\Pi)}$ and including the empty set as an isolated point.

2. Proofs

2.1. *On relative compactness in the Hausdorff metric*

In this section we record two general results, for later use, on the relationship between relative compactness and the Hausdorff metric. They are certainly known, but we were unable to locate suitable references within the literature. We make use of these results in the proof of Lemma 2.8.

Let M be any complete metric space and let $\mathcal{K}(M)$ be the set of compact subsets of M, equipped with the Hausdorff metric. Since both M and (consequently, also) $\mathcal{K}(M)$ are complete, within these spaces relative compactness is equivalent to total boundedness.

Lemma 2.1: *Suppose that K is a relatively compact subset of $\mathcal{K}(M)$. Then $\bigcup_{X\in K} X$ is a relatively compact subset of M.*

Proof: Since K is totally bounded, for each $\epsilon > 0$ there is a finite set $X_1,\dots,X_n$ of elements of $\mathcal{K}(M)$ such that, for any $X \in K$ there is some X_i such that $d_{\mathcal{K}(M)}(X, X_i) < \epsilon$. Let $Y = \bigcup_{i=1}^n X_i$ and note that $\bigcup_{X\in K} X \subseteq Y^{(\epsilon)}$. Since each X_i is compact in M, Y is also compact in M, and in particular Y is totally bounded. Hence also $\bigcup_{X\in K} X$ is totally bounded. □

Lemma 2.2: *Suppose that B is a relatively compact subset of M. Then $K = \{X \in \mathcal{K}(M)\,;\, X \subseteq B\}$ is a relatively compact subset of $\mathcal{K}(M)$.*

Proof: Since B is totally bounded, for all $\epsilon > 0$ there is a finite sequence $x_1,\dots,x_n$ such that each $x \in M$ is within distance ϵ from some x_i. For any non-empty compact subset X of B, if we set $X' = \{x_i\,;\, \exists x \in X, d_M(x, x_i) < \epsilon\}$, then $d_{\mathcal{K}(M)}(X, X') < \epsilon$. Thus, every such X is within distance ϵ in $\mathcal{K}(M)$ of one of the subsets of $\{x_1,\dots,x_n\}$. Since there are only finite many such subsets, this shows that K is totally bounded. □

2.2. *Relative compactness*

Our first objective is to characterize relative compactness in G, for which we follow a similar method to that employed in Section 3.6 of [5] for Skorohod spaces of càdlàg paths. We then transfer this characterization to Π and finally upgrade it to $\mathcal{K}(\Pi)$.

Recall that a step function is a piecewise constant function with finitely many discontinuities. Given a step function $g \in G$, let $(s_i(g))_{i=0}^m$ denote the (finite, ordered) sequence of times at which g is discontinuous, including also the two endpoints $s_0(g) = \sigma_g$ and $s_m(g) = 2$. With slight abuse of notation, extend $(s_i(g))$ to an infinite sequence $(s_i(g))_{i=0}^{\infty}$ by setting $s_i(g) = 2$ for all $i \geq m$.

Let $A_\delta \subseteq G$ be the set of step functions in G for which $\inf_i |s_i(g) - s_{i+1}(g)| > \delta$.

Lemma 2.3: *A_δ is a relatively compact subset of G.*

Proof: Let (g_n) be a sequence in A_δ. We will show that (g_n) has a convergent subsequence in G.

For all n, since $g_n \in A_\delta$ we have that the sequence $s_i(g_n) = 2$ for all $i \geq \delta^{-1}$. Noting that $[-1, 2]$ is compact, by diagonalization we may pass to a subsequence and assume without loss of generality that there exists $t_0, \ldots, t_{\lfloor \delta^{-1} \rfloor}, \ x_0, \ldots, x_{\lfloor \delta^{-1} \rfloor} \in [-1, 2]$ such that for $i \in \{0, \ldots, \lfloor \delta^{-1} \rfloor\}$

$$\lim_{n\to\infty} s_i(g_n) = t_i \qquad \text{and} \qquad \lim_{n\to\infty} g_n(s_i(g_n)) = x_i.$$

We define the function

$$g(t) = \begin{cases} x_i & \text{if } \ t_i \leq t < t_{i+1}, \ i = 0, \ldots, \lfloor \delta^{-1} \rfloor - 1 \\ x_{\lfloor \delta^{-1} \rfloor} & \text{if } \ t_{\lfloor \delta^{-1} \rfloor} \leq t \leq 2. \end{cases}$$

Then $g \in G$ and it is easily seen that $d_G(g_n, g) \to 0$. □

Lemma 2.4: *For each $g \in G$, it holds that $w'(g, \delta) \to 0$ as $\delta \to 0$.*

Proof: Let $g \in G$. For any $N \in \mathbb{N}$ define a sequence of times $(T_i^N)_{i=0}^{\infty}$, inductively, by setting $T_0^N = \sigma_g$ and

$$T_i^N = 2 \wedge \inf\{t \in (T_{i-1}^N, 2] : |g(t) - g(T_{i-1}^N)| > \tfrac{1}{N}\},$$

with the convention $\inf \emptyset = +\infty$.

Since g has left limits, the increasing sequence (T_i^N) has no limits points in $[-1, 2)$, and since g is constant on $[1, 2]$ there exists $k \in \mathbb{N}$ such that $T_{k-1}^N < 2 = T_k^N$. For all $0 < \delta < \min\{T_i^N - T_{i-1}^N : i \leq k\}$ we have $w'(g, \delta) < 2/N$, which completes the proof. □

Lemma 2.5: *Let $g_n, g \in G$ such that $d_G(g_n, g) \to 0$ as $n \to \infty$. Then, for all $\delta > 0$ we have*

$$\limsup_{n\to\infty} w'(g_n, \delta) \leq w'(g, \delta).$$

Proof: We have $d_G(g_n, g) \to 0$, so for each $n \in \mathbb{N}$ there exists $\lambda_n \in \Lambda[g_n, g]$ such that $\gamma_{g_n,g}(\lambda_n) \to 0$ and $d(g_n, g, \lambda_n) \to 0$. With notation as in (1.9), for any $\delta > 0$ we have

$$\begin{aligned} w'(g_n, \delta) &\le \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} |g(\lambda_n(s)) - g(\lambda_n(t))| + 2d(g_n, g, \lambda_n) \\ &\le \inf_{\{\lambda_n(t_i)\}} \max_i \sup_{s,t\in[\lambda_n(t_{i-1}),\lambda_n(t_i))} |g(s) - g(t)| + 2d(g_n, g, \lambda_n) \\ &= w'(g, \delta) + 2d(g_n, g, \lambda_n). \end{aligned} \tag{2.1}$$

Here, to deduce the first line we use the triangle inequality (comparing $g(s)$ to $g(\lambda_n(s))$ and similarly for t), and to deduce the second and third lines we use the definition of $\Lambda(g_n, g)$.

The result follows by letting $n \to \infty$ in (2.1). □

Lemma 2.6: *A subset A of G is relatively compact if and only if*

$$\lim_{\delta\to 0} \sup_{g\in A} w'(g, \delta) = 0. \tag{2.2}$$

Proof: Assume, first, that A is relatively compact. Suppose that (2.2) fails: then there exists $\eta > 0$, a sequence $\delta_n \to 0$ and a sequence (g_n) in G, such that $w'(g_n, \delta_n) \ge \eta$. By relative compactness of A and completeness of G, we may pass to a subsequence and assume, without loss of generality, that there exists $g \in G$ such that $d_G(g_n, g) \to 0$. Therefore, by (2.1) we have

$$\eta \le \limsup_{n\to\infty} w'(g_n, 1/n) \le \limsup_{n\to\infty} w'(g, 1/n)$$

which contradicts Lemma 2.4. Hence, in fact (2.2) must hold.

It remains to prove the converse. To this end, suppose that (2.2) holds, and we now look to show that A must be relatively compact. By (2.2), for each $N \in \mathbb{N}$ there exist δ_N such that

$$\sup_{g\in A} w'(g, \delta_N) \le N^{-1}.$$

Hence, by definition of w', for each $g \in A$ there exists a δ_N-sparse partition $\sigma_g = t_0 < t_1 < \cdots < t_m = 1 < t_{m+1} = 2$ such that

$$\max_i \sup_{s,t\in[t_{i-1},t_i)} |g(s) - g(t)| \le N^{-1}.$$

Recall the set A_δ from Lemma 2.3. We define a function $h \in A_{\delta_N}$, where $h : [\sigma_g, 2] \to [-1, 1]$, by

$$h(t) = \begin{cases} g(t_{i-1}) & \text{if } t \in [t_{i-1}, t_i) \text{ for some } i = 1, \ldots, m \\ g(t_m) & \text{if } t \in [1, 2]. \end{cases}$$

Take λ to be the identity function on $[\sigma_g, 2]$ and note that $\lambda \in \Lambda[g,h]$ with $\gamma_{g,h} = 0$. By definition of h we have $d(g,h,\lambda) \le N^{-1}$, and hence $\rho(g,h) < N^{-1}$. Thus $A \subseteq A_{\delta_N}^{(N^{-1})}$ where

$$A_{\delta_N}^{(N^{-1})} = \left\{ f \in G : \rho(f', f) < N^{-1} \text{ for some } f' \in A_{\delta_N} \right\}$$

is the expansion of A_{δ_N} by radius N^{-1}.

Since G is complete, within G relative compactness is equivalent to total boundedness. Hence, by Lemma 2.3, A_{δ_N} is totally bounded. Since $A \subseteq \bigcap_N A_{\delta_N}^{(N^{-1})}$, A is also totally bounded, which completes the proof. □

Lemma 2.7: *A subset B of Π is relatively compact if and only if, for all bounded $\Lambda_{L,T}$*

$$\lim_{\delta \to 0} \sup_{f \in B} w'(f, \delta, \Lambda_{L,T}) = 0. \tag{2.3}$$

Proof: From (1.21) and Lemma 2.6, we have that $B \subseteq \Pi$ is relatively compact if and only if

$$\lim_{\delta \to 0} \sup_{f \in B} w'(\bar{f}, \delta) = 0. \tag{2.4}$$

Recall the re-scaling of time $\kappa_t = \tanh^{-1} t$ is a homeomorphism between $[-1,1]$ and $[-\infty,\infty]$, and note that κ_t is bi-Lipschitz on closed intervals within $(-1,1)$. It can be seen that the map (1.1) is bi-Lipschitz in a similar sense: for any bounded box $\Lambda_{L,T}$ there exists $C \in (0,\infty)$ such that for all $f \in \Pi$ and all $s,t \in [\sigma_f, \infty]$,

$$|\bar{f}(s) - \bar{f}(t)| \le |f_{L,T}(\kappa_s) - f_{L,T}(\kappa_t)| \le C|\bar{f}(s) - \bar{f}(t)|. \tag{2.5}$$

Proving the inequality (2.5) is elementary but cumbersome, and we omit the argument in the interests of brevity. However, with (2.5) in hand, it is immediate from (1.9) and (1.12) that $w'(\bar{f},\delta) \le w'(f,\delta,\Lambda_{L,T}) \le Cw'(\bar{f},\delta)$ and thus (2.3) is equivalent to (2.4). □

Lemma 2.8: *A subset C of $\mathcal{K}(\Pi)$ is relatively compact if and only if, for all bounded $\Lambda_{L,T}$,*

$$\lim_{\delta \to 0} \sup_{B \in C} \sup_{f \in B} w'(f, \delta, \Lambda_{L,T}) = 0. \tag{2.6}$$

Proof: First, assume that $C \in \mathcal{K}(\Pi)$ is relatively compact. Then, by Lemma 2.1, the union $B' = \bigcup_{B \in C} B$ is a relatively compact subset of Π. Equation (2.6) follows by applying Lemma 2.7.

It remains to prove the converse. To this end, assume that $C \in \mathcal{K}(\Pi)$ and (2.6) holds. Again, consider the union $B' = \bigcup_{B\in C} B$, and note that now Lemma 2.7 implies that B' is a relatively compact subset of Π. Hence, by Lemma 2.2, the set K' of compact subsets of B' is a relatively compact subset of $\mathcal{K}(\Pi)$. Since $C \subseteq K'$, it follows that C is relatively compact. □

2.3. *Tightness*

As in Section 2.2, if X is a subset of some metric space $\mathbb{X}$, we write $A^{(\epsilon)}$ for the expansion of the set A by radius $\epsilon > 0$.

Recall that a sequence of random variables (X_n) taking values in some metric space $\mathbb{X}$ is said to be tight if for all $\epsilon > 0$ there exists a (deterministic) compact subset K of $\mathbb{X}$ such that $\inf_n \mathbb{P}[X_n \in K] \geq 1 - \epsilon$.

In fact, by combining Lemma 2.2.1 and Theorem 2.2.2 of [5], for a complete and separable metric space $\mathbb{X}$ tightness of $(\mathcal{X}_n)$ is equivalent to asking that, for all $\epsilon > 0$ there exists a compact subset K of $\mathbb{X}$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left[X_n \in K^{(\epsilon)}\right] \geq 1 - \epsilon. \tag{2.7}$$

We now provide proofs of Propositions 1.2 and 1.3.

Proof: [Of Proposition 1.2.] First, suppose that $(\mathcal{X}_n)$ is tight in $\mathcal{K}(\Pi)$. That is, for each $\epsilon > 0$ there exists a compact set K of $\mathcal{K}(\Pi)$ such that $\liminf_n \mathbb{P}[\mathcal{X}_n \in K] \geq 1 - \epsilon$. By Lemma 2.8, for any $\eta > 0$ we can choose δ_0 such that for all $\delta \in (0, \delta_0)$ we have $\sup_{B\in K} \sup_{f\in B} w'(f, \delta, \Lambda_{L,T}) < \eta$, and thus

$$\liminf_{n\to\infty} \mathbb{P}\left[\sup_{f\in\mathcal{X}_n} w'(f, \delta, \Lambda_{L,T}) \leq \eta\right] \geq 1 - \epsilon.$$

Equation (1.14) follows.

It remains to prove the converse. Suppose instead that $(\mathcal{X}_n)$ satisfies (1.14). Our plan is to show that $\mathcal{X}_n$ is likely to be close to a suitably chosen set of step functions.

For $\delta > 0$, let B_δ be the set of step functions in Π with jumps spaced at least δ apart (in time). It is easily seen, from Lemma 2.7 that B_δ is a relatively compact subset of Π. By Lemma 2.2, $C_\delta = \{B \in \mathcal{K}(\Pi)\,;\, B \subseteq \overline{B_\delta}\}$ is a relatively compact subset of $\mathcal{K}(\Pi)$.

Fix $L, T \in (0, \infty)$ and let $\eta > 0$. Consider first some f and δ such that $w'(f, \delta, \Lambda_{L,T}) \leq \eta$. We define a function $f' \in B_\delta$ as follows. Since $w'(f, \delta, \Lambda_{L,T}) \leq \eta$ there exists a δ-sparse partition (t_i) of $[(-T) \vee \sigma_f, T]$

such that $\max_i \sup_{s,t\in[t_i,t_{i+1})} |f_{L,T}(s) - f_{L,T}(t)| \leq \eta$. Set

$$f'(t) = \begin{cases} f_{L,T}(t_i) & \text{for } t \in [t_i, t_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sup_{|t|\leq T} |f_{L,T}(t) - f'(t)| \leq \eta$. We have

$$\begin{aligned} \sup_{t\in[-1,2]} |\bar{f}(t) - \bar{f}'(t)| &\leq \sup_{t\in[-1,2]} |\bar{f}(t) - \overline{f_{L,T}}(t)| + \sup_{t\in[-1,2]} |\overline{f_{L,T}}(t) - \bar{f}'(t)| \\ &\leq \left(|1 - \tanh(L)| + \frac{2}{1+|T|}\right) + \left(\eta + \frac{2}{1+|T|}\right). \end{aligned}$$

Here, to deduce the second inequality, we split each sup into $|\kappa_t| \leq T$ and $|\kappa_t| > T$, and then use the left hand side of (2.5) along with (1.2). Choosing T, L sufficiently large (dependent on η but not on δ) we obtain that $\sup_t |\bar{f}(t) - \bar{f}'(t)| \leq 2\eta$, which in turn implies that $d_\Pi(f, f') = d_G(\bar{f}, \bar{f}') \leq 2\eta$. To summarise:

$$w'(f, \delta, \Lambda_{L,T}) \leq \eta \qquad \Rightarrow \qquad \exists f' \in B_\delta \text{ such that } d_\Pi(f, f') \leq 2\eta. \tag{2.8}$$

We are now in a position to complete the proof.

Let $\eta, \epsilon > 0$ and let $\Lambda_{L,T}$ be large enough that (2.8) holds. By (1.14) choose $\delta > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left[\sup_{f\in\mathcal{X}_n} w'(f, \delta, \Lambda_L, T) \leq \eta\right] \geq 1 - \epsilon. \tag{2.9}$$

Consider now the event that $\sup_{f\in\mathcal{X}_n} w'(f, \delta, \Lambda_{L,T}) \leq \eta$, and let $\mathcal{X}'_n = \{f' \,;\, f \in \mathcal{X}_n\}$, where f' is the function given in (2.8). Since $\mathcal{X}'_n \subseteq B_\delta$ we have $\overline{\mathcal{X}'_n} \in C_\delta$. Moreover, by (2.8) the Hausdorff distance between $\mathcal{X}_n$ and $\overline{\mathcal{X}'_n}$ is bounded above by 2η. Hence, on this event $\mathcal{X}_n \in C_\delta^{(2\eta)}$, where $C_\delta^{(2\eta)}$ denotes the expansion (in the Hausdorff metric) of C_δ by radius 2η; thus also $\mathcal{X}_n \in \overline{C_\delta}^{(2\eta)}$. Thus, from (2.9) we obtain

$$\liminf_{n\to\infty} \mathbb{P}\left[\mathcal{X}_n \in \overline{C_\delta}^{(2\eta)}\right] \geq 1 - \epsilon.$$

Take $\eta = \epsilon/2$. Noting that $\overline{C}_\delta$ is compact in $\mathcal{K}(\Pi)$, by (2.7) we have that $(\mathcal{X}_n)$ is tight. □

Proof: [Of Proposition 1.3.] Take $f \in \Pi$, and fix $\delta > 0$ and $\Lambda_{L,T}$. We aim to relate $w'(f, \delta, \Lambda_{L,T})$ to $w''(f, \delta, \Lambda_{L,T})$ and $w^+(f, \delta, \Lambda_{L,T})$. The argument is similar to the proof of Theorem 12.4 in [3].

To ease our notation, for the duration of this proof we introduce the shorthand $w' = w'(f, \delta, \Lambda_{L,T})$, similarly for w'', w^+ and also w_i from (1.13).

Let us also write $h = f_{L,T}$, and note that $\sigma_f = \sigma_h$. Without loss of generality we may assume that $\sigma_h < T$, since h is constant outside of time $[-T, T]$, and no contribution to w', w'', w^+ is made at such times.

Take α such that $w' < \alpha$. Then there exists a δ-sparse partition (s_i) of $[\sigma_h, T]$ such that, for all i, $w_i \le \alpha$. If s, t are such that $|s-t| < \delta$ then, there exists at most one value of i such that $s_i \in [s, t)$. Therefore, if $s < u < t$, we have $s, u \in [s_i, s_{i+1})$ for some i, or $t, u \in [s_i, s_{i+1})$ for some i, which means that either $|h(s) - h(u)| < \alpha$ or $|h(u) - h(t)| < \alpha$. Thus $w'' \le \alpha$, and since α was arbitrary we have $w'' \le w'$.

Since the first element of the δ-sparse partition (s_i) is $s_1 = 0$, and $s_2 - s_1 > \delta$, we have that $w^+ \le w'$. Thus,

$$w'' \vee w^+ \le w', \tag{2.10}$$

and it remains to prove a suitable inequality in the reverse direction. The reader may wish to glance ahead at (2.12) to see where we are heading.

Our first step is to show that for all $t_1 \le s \le t \le t_2$ such that $|t_2 - t_1| \le \delta$ we have

$$\big|h(t_1) - h(s)\big| \wedge \big|h(t) - h(t_2)\big| \le 2w''. \tag{2.11}$$

Equation (2.11) is easily seen: if $|h(t_1) - h(s)| \ge w''$ then (by definition of w'') we have both $|h(s) - h(t)| \le w''$ and $|h(s) - h(t_2)| \le w''$, which implies that $|h(t) - h(t_2)| \le 2w''$.

Now take any $\alpha > w'' \vee w^+$. Suppose that h has at a pair of jumps, each of magnitude exceeding 2α, at the points $u_1 < u_2$. If $|u_2 - u_1| < \delta$ then we can find disjoint intervals $(t_1, s) \ni u_1$ and $(t, t_2) \ni u_2$ with $|t_1 - t_2| < \delta$, and this would contradict (2.11). Note also that w^+ controls movement during $[\sigma_h, \sigma_h + \delta)$. We thus have that:

(1) Any two jumps of h of magnitude exceeding 2α must occur at least δ time apart;
(2) h has no jumps of magnitude exceeding 2α during $[\sigma_h, \sigma_h + \delta)$;
(3) h is constant outside of $[-T, T]$.

Therefore, there exists a finite sequence (s_i) with $\sigma_f = s_0 < s_1 < \ldots < s_m = T$, with $s_{i+1} - s_i > \delta$ for all i, which contains all points at which h jumps more than 2α. If $s_{i+1} - s_i > \delta$ for any pair of adjacent points, enlarge the (s_i) by including their midpoint; continue doing so inductively until we have an enlarged partition satisfying, for all i,

$$\frac{\delta}{2} < s_{i+1} - s_i \le \delta.$$

Thus, (s_i) forms a $\delta/2$-sparse partition of $[\sigma_f, T]$, and that all the jumps of h with magnitude exceeding 2α occur at some s_i.

We now look to control w', using the partition (s_i). Fix i and consider t_1, t_2 such that $s_i \le t_1 < t_2 < s_{i+1}$, which implies $|t_1 - t_2| < \delta$. Let

$$\sigma_1 = \sup\{\sigma \in [t_1, t_2]\,;\, |h(t_1) - h(\sigma)| \le 2\alpha\},$$
$$\sigma_2 = \inf\{\sigma \in [t_1, t_2]\,;\, |h(\sigma) - h(t_2)| \le 2\alpha\}.$$

If $\sigma_1 < \sigma_2$ then we have s, t such that $t_1 \le \sigma_1 < s < t < \sigma_2 \le t_2$, which contradicts (2.11). Thus $\sigma_2 \le \sigma_1$, which implies that $|h(t_1) - h(\sigma-)| \le 2\alpha$ and (by right continuity of h) that $|h(\sigma) - h(t_2)| \le 2\alpha$. Since $\sigma_1 \in [s_i, s_{i+1})$ the jump at σ_1 has magnitude at most 2α. Thus $|h(t_1) - h(t_2)| \le 6\alpha$.

From the previous paragraph we have $w_i(f, \delta/2, \Lambda_{l,T}) \le 6\alpha$, and since i was arbitrary we have $w'(f, \delta/2, \Lambda_{L,T}) \le 6\alpha$. Since α was also arbitrary we have

$$w'(f, \delta/2, \Lambda_{L,T}) \le 6\Big(w''(f, \delta, \Lambda_{L,T}) \vee w^+(f, \delta, \Lambda_{L,T})\Big). \tag{2.12}$$

Combining (2.10) and (2.12), we have that (1.14) is equivalent to (1.18), so Proposition 1.3 follows from Proposition 1.2. □

Remark 2.9: Equations (2.10) and (2.12), when applied to Lemmas 2.7 and 2.8, yield characterizations of relative compactness in Π and $\mathcal{K}(\Pi)$ based on w'' and w^+. This is left for the reader.

3. On the failure of tightness due to large jumps

In this section we will briefly outline why a system of coalescing random walkers, making suitably heavy tailed jumps, will fail to be tight in $\mathcal{K}(\Pi)$ under the space-time scaling limit that would correspond to a single random walker converging to an α-stable Lévy process. Heuristically, the presence of too many particles at once results in at least one of them making a large movement too soon after its own starting time; tightness fails as a consequence, in both $\mathcal{K}(\Pi)$ and $\mathcal{K}(\widetilde{\Pi})$.

More precisely, consider a random sequence $(\mathcal{X}_n)$ of sets of paths. If $(\mathcal{X}_n)$ consists of continuous paths and is tight in $\mathcal{K}(\widetilde{\Pi})$, then (1.6) holds and (using (1.11)) it is easily seen that for all bounded $\Lambda_{L,T}$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}\left[\sup_{f\in\mathcal{X}_n} w^+(f, \delta, \Lambda_{L,T}) > \eta\right] = 0. \tag{3.1}$$

Alternatively, if $(\mathcal{X}_n)$ consists of càdlàg paths and tightness holds in $\mathcal{K}(\Pi)$, then (1.14) holds and (3.1) is a trivial consequence. Thus, for both $\mathcal{K}(\Pi)$

and $\mathcal{K}(\widetilde{\Pi})$, tightness necessitates (3.1), which we might describe in words as uniform right continuity in probability at starting times.

Let us now restrict ourselves to the example of a collection $\mathcal{Y}$ of càdlàg paths of continuous time coalescing random walks on the space-time lattice $\mathbb{Z} \times \mathbb{R}$. More precisely, let J be a $\mathbb{Z}$ valued random variable, and specify that if a random walk is at location $x \in \mathbb{Z}$, then after an exponential time with rate 1 the random walker jumps to $x + J$. All such jumps take an independent copy of J, and all random walkers coalesce instantaneously upon meeting, leaving at most one random walker at each point of space-time. We write $f_{(y,s)} : [s, \infty) \to \mathbb{Z}$ for the càdlàg path of the random walker beginning at $(y, s) \in \mathbb{Z} \times \mathbb{R}$, and thus $\mathcal{Y} = \{f_{(y,s)}\,;\,(y, s) \in \mathbb{Z} \times \mathbb{R}\}$.

Let $\alpha \in (0, 2]$. Following [1], which treats only the case $\alpha = 2$ of a diffusive rescaling, let $D_n(\eta, \delta)$ be the event that one of the paths within $\mathcal{Y}$ has a jump which originates within $[\eta n, \infty) \times [0, \delta n^\alpha]$ and lands within the negative half-plane $(-\infty, 0] \times [0, \delta n^\alpha]$.

Let $\mathcal{Y}_n$ denote $\mathcal{Y}$ with time sped up by a factor n^α and space compressed by a factor n. Thus, the region $[0, \eta] \times [0, \delta]$ of rescaled space-time (for $\mathcal{Y}_n$) corresponds to the unscaled region $[0, \eta n] \times [0, \delta n^\alpha]$ (for $\mathcal{Y}$).

Lemma 3.1: *If* (3.1) *holds for* $(\mathcal{Y}_n)$ *then*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}\left[D_n(\eta, \delta)\right] = 0. \tag{3.2}$$

Proof: Without loss of generality, suppose that $T \geq \delta$ and $L \geq \eta$. Suppose that the event $D_n(\eta, \delta)$ occurs. We seek to show that $\sup_{f \in \mathcal{Y}_n} w^+(f, \delta, \Lambda_{L,T}) > \eta$.

We have that there is some path $f \in \mathcal{Y}$ which begins during $[0, \delta n^\alpha]$ and makes a jump originating from within $[\eta n, \infty) \times [0, \delta n^\alpha]$ that lands within $(-\infty, 0] \times [0, \delta n^\alpha]$. After rescaling f corresponds to some path in $\mathcal{Y}_n$ which has starting (rescaled) time within $[0, \delta]$ and makes a jump of (rescaled) magnitude at least η before time δ. This completes the proof. □

Remark 3.2: In the proof of Theorem 1.1 in [1], which treats the diffusive case $\alpha = 2$ and uses with linear interpolation in place of jumps, it was shown that (3.2) failed if the $3 - \epsilon$ moment of J was infinite for any $\epsilon > 0$. From this, [1] deduced that random walk 'approximations' to the Brownian web could be tight in $\mathcal{K}(\widetilde{\Pi})$ only if J had at least $3 - \epsilon$ moments.

Let us now consider $\alpha \in (0, 2)$. For clarity, we will treat only the special case in which $p(j) = \mathbb{P}[J = j] = \mathbb{P}[J = -j] \sim Cj^{-\alpha-1}$, where C is a normalizing constant. Therefore, each of our random walkers is, individually, within the domain of attraction of an α-stable Lévy process. (Note that this assumption is sufficient to ensure that our system of coalescing random walkers on $\mathbb{Z} \times \mathbb{R}$ is well defined.)

Let $F(k) = \sum_{j=k}^{\infty} p(j)$. Random walk jumps that start within $[\eta n, \infty)$ and land in $(-\infty, 0]$ occur within $\mathcal{Y}$ at rate $\sum_{k=\eta n}^{\infty} F(k)$. Thus,

$$\mathbb{P}[D_n(\eta, \delta)] = 1 - \exp\left(-\delta n^{\alpha} \sum_{k=\eta n}^{\infty} F(k)\right).$$

Hence, (3.2) holds if and only if

$$\limsup_{n\to\infty} n^{\alpha} \sum_{k=\eta n}^{\infty} F(k) < \infty. \tag{3.3}$$

Recall that $p(j) \sim Cj^{-\alpha-1}$, which means that $F(k) \sim C'k^{-\alpha}$. Hence, the argument of the lim sup on the left hand side of equation (3.3) is of order $C'n^{\alpha}(\eta n)^{-\alpha+1}\frac{1}{\alpha+1} = \mathcal{O}(n)$, and thus (3.3) fails. Hence, by Lemma 3.1 and Proposition 1.3, tightness in $\mathcal{K}(\Pi)$ fails.

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